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Application of nonlinear filtering to credit risk

Vivek S. Borkar [a], Mrinal K. Ghosh [b,*], G. Rangarajan [b]

[a] School of Technology and Computer Science, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India

[b] Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India

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ABSTRACT

Merton's model views equity as a call option on the asset of the firm. Thus the asset is partially observed through the equity. Then using nonlinear filtering an explicit expression for likelihood ratio for underlying parameters in terms of the nonlinear filter is obtained. As the evolution of the filter itself depends on the parameters in question, this does not permit direct maximum likelihood estimation, but does pave the way for the 'Expectation–Maximization' method for estimating parameters.

1. Introduction

The assessment of credit risk is one of the most important problems in quantitative finance. A powerful approach to this is based on the option theoretic interpretation by Merton [27] (also see [4]). This approach is referred to as the *Asset Value Model* (AVM), or *structural approach to credit risk* [2,5,14,16]. While theoretically very gratifying, this still leaves wide open several computational issues. The main aim of this article is to propose a computational scheme for credit risk evaluation based on AVM. While for purposes of exposition we stick to a simple model, the underlying philosophy is broader and can be extended to more elaborate models. It has the advantage of having a rigorous footing based on methodologies that have already been utilized extensively in the signal processing community and it accounts for aspects not addressed hitherto in existing literature, as will become apparent. There is another approach to credit risk known as the reduced form (or intensity based) approach where the reason behind a default is not investigated. Instead, the dynamics of default are exogenously given through a default rate or intensity; see [1,14,16] and the references therein. We do not follow this approach in this paper.

We begin by recalling in some detail the AVM model and the current status of this problem. In this approach, the asset value process $\{A_t\}$ of the firm is assumed to follow a geometric Brownian motion (GBM) given by

$$dA_t = \nu A_t dt + \sigma A_t dW_t, \quad t \geq 0 \tag{1.1}$$

where ν is the net mean return rate on the assets, i.e., $\nu = \mu - \gamma$, where μ is the gross mean return on the assets and γ is the proportional cash payout rate; σ is the volatility, and $\{W_t\}$ is the standard Brownian motion. It is also assumed that the company has a simple *capital structure* consisting of one *debt obligation* and one type of *equity*. Let $\mathcal{E}_t$ denote the equity process of the company which is traded publicly. Suppose the process $\mathcal{D}_t$ denotes the market value of the debt obligation of the company which is assumed to have the *cash profile* of a *zero-coupon bond* maturing at a prescribed future time T and interest adjusted *face value* K. In the classical model [27] the company defaults if $A_T < K$. If the company defaults, then the payoff to the equity holders is zero. If it does not, i.e., $A_T \geq K$, then there is a net profit of $A_T - K$ after paying back the debt. Thus the total payoff to equity holders is $(A_T - K)^+ \stackrel{\text{def}}{=} \max(A_T - K, 0)$, which is identical to the payoff for a European call option on $\{A_t\}$ with strike price K, constant dividend rate γ and maturity T. Therefore for $t \in [0, T]$, $\mathcal{E}_t$ is a long European call C_t^γ from the point of view of the equity holders. Thus by the Black–Scholes–Merton option pricing formula it follows that

$$\mathcal{E}_t = C_t^\gamma = e^{-\gamma(T-t)} A_t \Phi(d_1(A_t, T-t)) - K e^{-r(T-t)} \Phi(d_2(A_t, T-t)) \tag{1.2}$$

where, for $r \stackrel{\text{def}}{=}$ the risk-free interest rate,

$$d_1(x, t) \stackrel{\text{def}}{=} \frac{\log\left(\frac{x}{K}\right) + \left(r - \gamma + \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{t}} \tag{1.3}$$

* Corresponding author.
E-mail addresses: borkar@tifr.res.in (V.S. Borkar), mkg@math.iisc.ernet.in (M.K. Ghosh), rangaraj@math.iisc.ernet.in (G. Rangarajan).

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$$d_2(x,t) \stackrel{\text{def}}{=} \frac{\log\left(\frac{x}{K}\right) + \left(r - \gamma - \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{t}}, \tag{1.4}$$

and $\Phi(\cdot)$ as usual denotes the Gaussian distribution function. The value of the debt obligation D_T at time T is given by

$$\mathcal{D}_T = \min(K, A_T) = K - (K - A_T)^+.$$

The above payoff is equivalent to that of a portfolio consisting of a default-free loan with face value K maturing at T and a short European put option on $\{A_t\}$ with strike price K and maturity T. Thus the value of $\mathcal{D}_t$ at time t is given by

$$\mathcal{D}_t = K e^{-r(T-t)} - P_t^\gamma \tag{1.5}$$

where P_t^γ denotes the price of the put option on A_t with strike price K, constant dividend rate γ and maturity T. Using the *put-call parity*

$$A_t e^{-\gamma(T-t)} + P_t^\gamma = C_t^\gamma + K e^{-r(T-t)}$$

we obtain

$$\mathcal{D}_t = A_t e^{-\gamma(T-t)} - \mathcal{E}_t \tag{1.6}$$

where A_t and $\mathcal{E}_t$ are determined from (1.1) and (1.2) respectively. The Eq. (1.6) gives the '*theoretical*' price of the debt at time t.

Another key concept in the AVM is the default probability. In the classical model the conditional probability of default is given by

$$P(A_T < K \mid A_t) = \Phi\left(\frac{\log L_t - m(T-t)}{\sigma\sqrt{T-t}}\right), \tag{1.7}$$

where $m = \nu - \frac{1}{2}\sigma^2$ and $L_t = \frac{K}{A_t}$ is the leverage ratio of the firm at time t.

We have thus far considered the case when default occurs only at the time T of maturity of the debt. Black and Cox [3] introduced the concept of *first passage time* to compute the default probability. In this model, default occurs at a random time $\tau \in (0, T]$ when the asset value A_t falls below a level D for the first time. We assume that $D \le K$. If $D > K$, then the debt holders are fully protected [16]. More precisely, let

$$\tau_1 = \begin{cases} T & \text{if } A_T < K \\ \infty & \text{otherwise.} \end{cases}$$

Let τ_2 be the stopping time given by

$$\tau_2 = \inf\{t \in (0, T] \mid A_t < D\}.$$

Then the default time τ is given by

$$\tau = \tau_1 \wedge \tau_2.$$

Thus the forward conditional default probability at time t is given by

$$p_d(A_t) = 1 - P(\tau_1 \wedge \tau_2 > T | A_t).$$

A simple computation shows that

$$p_d(A_t) = \Phi\left(\frac{\log L_t - m(T-t)}{\sigma\sqrt{T-t}}\right) + \left(\frac{D}{A_t}\right)^{\frac{2m}{\sigma^2}} \Phi\left(\frac{\log(D^2/(KA_t)) + m(T-t)}{\sigma\sqrt{T-t}}\right). \tag{1.8}$$

This default is obviously higher than the corresponding default probability in the classical approach. Note that (1.7) is obtained as a special case of (1.8) with $D = 0$.

In the first passage model the payoff to equity holders at maturity is given by

$$\mathcal{E}_T = (A_T - K)^+ I\{M_T \ge D\}$$

where $M_t = \min_{s \le t} A_s$. The above payoff corresponds to a European down-and-out call on A_t with strike price K, barrier $D(< K)$, constant dividend rate γ and maturity T. Thus at an earlier time t, $\mathcal{E}_t$ is given by

$$\mathcal{E}_t = C_t^\gamma - e^{-\gamma(T-t)} A_t \left(\frac{D}{A_t}\right)^{\frac{2(r-\gamma)}{\sigma^2}+1} \Phi(d_3(A_t, T-t)) + K e^{-r(T-t)} \left(\frac{D}{A_t}\right)^{\frac{2(r-\gamma)}{\sigma^2}-1} \Phi(d_4(A_t, T-t)) \tag{1.9}$$

where

$$d_3(x,t) \stackrel{\text{def}}{=} \frac{\log\left(\frac{D^2}{Kx}\right) + \left(r - \gamma + \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{t}} \tag{1.10}$$

$$d_4(x,t) \stackrel{\text{def}}{=} \frac{\log\left(\frac{D^2}{Kx}\right) + \left(r - \gamma - \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{t}}. \tag{1.11}$$

In this model, the value of the debt obligation $\mathcal{D}_T$ at time T is given by

$$\mathcal{D}_T = K - (K - A_T)^+ + (A_T - K)^+ I\{M_T < D\}$$

which is equivalent to a portfolio consisting of a risk free loan with face value K, a short European put on A_t with strike price K, constant dividend rate γ and maturity T, and a long European down-in-call on A_t with strike price K, dividend rate γ, barrier D and maturity T. Therefore at an earlier time the value of the debt $\mathcal{D}_t$ is given by

$$\mathcal{D}_t = A_t e^{-\gamma(T-t)} - C_t^\gamma + e^{-\gamma(T-t)} A_t \left(\frac{D}{A_t}\right)^{\frac{2(r-\gamma)}{\sigma^2}+1} \times \Phi(d_3(A_t, T-t)) - K e^{-r(T-t)} \left(\frac{D}{A_t}\right)^{\frac{2(r-\gamma)}{\sigma^2}-1} \times \Phi(d_4(A_t, T-t)). \tag{1.12}$$

Various extensions of the first passage time models have been studied in the literature which in particular include the case when the default boundary is given by a suitable stochastic process; see [2] and the references therein. The tractability of more general models declines rapidly with growing enrichment of the models, as pointed out in [13]. The AVM is the theoretical basis for the popular commercial *estimated default frequency* (EDF) by KMV, *default probabilities* by Moody's and related ratings—see [14,22]. But these are based on historical data used in their commercial software. These procedures are proprietary and not available in the public domain. The option theoretic AVM models have also become an integral part of valuations of corporate debts using (1.2) and (1.5). One of the major difficulties in this approach is that the asset value process $\{A_t\}$ is not observable and the parameters ν, σ are unknown. Since the equity process $\{\mathcal{E}_t\}$ is traded in the market, it is therefore observable. Suppose we assume that $\{\mathcal{E}_t\}$ is also a GBM given by, say,

$$d\mathcal{E}_t = \mu_E \mathcal{E}_t dt + \sigma_E \mathcal{E}_t dW_t' \tag{1.13}$$

where $\{W_t'\}$ is a standard Brownian motion. Since $\{\mathcal{E}_t\}$ is observable, the parameters μ_E, σ_E can be estimated from the market data. Assuming $\gamma = 0$, since $\{\mathcal{E}_t\}$ is a call option on $\{A_t\}$, using Ito's formula and some additional analysis, it has been shown in [5] that

$$\frac{\sigma_E}{\sigma} = \frac{A_t}{\mathcal{E}_t} \Phi(d_1(A_t, T-t)). \tag{1.14}$$

Now A_t and σ are determined from the Eqs. (1.2) and (1.14). This is the standard textbook approach to valuing corporate