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A reflected diffusion process in a regime-switching environment

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Abstract

This paper provides steady-state analysis of a reflected diffusion process governed by a regime-switching environment. We characterize differential equations satisfied by the limiting densities for overlapped and non-overlapped cases of reflecting boundary positions. Also we provide closed-form solutions for some specific cases and propose numerical methods for general cases.

Keywords: Brownian motion; Diffusion process; Reflecting boundary; Random environment; Steady-state density; Fluid flow model

1. Introduction

Recently, there has been a substantial interest in studying the steady-state density of a reflected diffusion process which arise in a queueing model or a financial market model. For example, Karandikar and Kulkarni [3] constructed a stochastic fluid model whose buffer content process is a Brownian motion with a reflecting boundary at zero, and investigated the properties of the steady-state of a $D|D|1$ queue or a Asynchronous Transfer Mode (ATM) network. They modelled the external environment as a continuous-time Markov-switching process, switching on a finite state space. Ang and Barria [1] presented a second-order fluid flow model of a queue with finite capacity buffer and a variable net input process changing according to a finite-state, continuous-time Markov process with two reflecting barriers. Linetsky [5] studied the analytical representation of transition densities for reflected *one-dimensional* diffusions and provided explicit expressions of the densities for some cases.

Continuing this line of research, in this paper we study the steady-state density of a general two-state Markov-switching diffusion process reflected at the two changing boundaries. Previous research has mostly focused on the case where the drift and volatility of the diffusion changes with constant boundaries; boundaries stay fixed regardless of the changes of the underlying Markov process. An important feature of our model is that the diffusion process jumps to the nearer one of the new reflecting boundaries when the state of the Markov chain changes and the diffusion process is out of the new reflection boundaries. This type of model appears in Jang et al. [2] which explores investor's behavior in a financial market in the presence of transaction costs when regime switching occurs in the stock price process. However, they consider only the case where the diffusing intervals of the two regimes do not overlap, whereas in this paper we study non-overlapped case as well as the overlapped case.

We characterize differential equations satisfied by the limiting densities for the two cases of reflecting boundary positions. We provide closed-form solutions when the diffusion process is an arithmetic or geometric Brownian motion and propose numerical methods for general processes.

2. The model

Consider a reflected diffusion process $\{Z(t), t \geqslant 0\}$ in a regime-switching environment. We assume that the environment has two regimes, 0 and 1, and the regime $i \in \{0, 1\}$ switches into regime $j \in \{0, 1\} (j \neq i)$ with rate $\lambda_i > 0$ independently of the diffusion process. We denote the dynamics of the environment by continuous time Markov process $\{I(t), t \geqslant 0\}$ taking values in the set $\{0, 1\}$, hence the environment stays in regime $I(t)$ at time t.

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The diffusion process evolves as

$$\mathrm{d}Z(t) = \mu_{I(t)}(Z(t))\,\mathrm{d}t + \sigma_{I(t)}(Z(t))\,\mathrm{d}W_t \quad \text{for } \underline{Z}_{I(t)} \leqslant Z(t) \leqslant \bar{Z}_{I(t)},$$

where $\{W_t, t \geqslant 0\}$ is a standard Brownian motion and $\underline{Z}_{I(t)}$ and $\bar{Z}_{I(t)}$ are the lower and upper reflection boundaries in regime $I(t)$. We assume the drift function $\mu_{I(t)}(\cdot)$ and the volatility function $\sigma_{I(t)}(\cdot)$ are continuously twice differentiable functions for each $I(t)$. If $I(t-) \neq I(t)$ and $Z(t-) < \underline{Z}_{I(t)}(Z(t-) > \bar{Z}_{I(t)})$, then $Z(t)$ becomes $\underline{Z}_{I(t)}(\bar{Z}_{I(t)}$, resp.). In other words, if the environment changes from regimes i to j at time t and $Z(t-)$ is not located on the interval $[\underline{Z}_j, \bar{Z}_j]$, $Z(t-)$ immediately jumps to one of the boundaries in the regime j nearer to $Z(t-)$. Assume also $\sigma_i(z) > 0$ for $z \in [\underline{Z}_i, \bar{Z}_i]$ and $i \in \{0, 1\}$.

The model described above is quite general compared with other models, for example, the second-order fluid flow model in [3], in the sense that the diffusion process has a stochastic drift and volatility, and the reflecting boundaries change according to the external environment.

Also a direct application of the model can be found in the field of finance. More specifically, Jang et al. [2] found that investor's optimal behavior in a financial market in the presence of transaction costs can be modelled like this if regime switching occurs in the stock price process. We do not present the details since its set-up requires far more financial background.

The following sections provide us some results of calculating the limiting densities for the two cases: the cases with non-overlapped intervals and overlapped intervals. We confirm that the analysis in the following sections is directly applicable to other cases, for example, the case with equal reflecting boundaries or the case where one interval includes the other.

3. The case with non-overlapped intervals

In this section, we explore the case where the diffusion intervals are not overlapped, that is,

$$\underline{Z}_0 < \bar{Z}_0 \leqslant \underline{Z}_1 < \bar{Z}_1 < \infty.$$

3.1. Analysis on steady-state

We can easily check that $\{(Z(t), I(t)), t \geqslant 0\}$ is a Markov process on the state space $([\underline{Z}_0, \bar{Z}_0] \times \{0\}) \bigcup ([\underline{Z}_1, \bar{Z}_1] \times \{1\})$. Let f be a real-valued function from $(-\infty, \infty) \times \{0, 1\}$ such that $f(\cdot, i)$ is continuously twice differentiable for each $i \in \{0, 1\}$. Also define the operator $\mathscr{U}_t$ for $t \geqslant 0$ by

$$(\mathscr{U}_t f)(z, j) = E[f(Z(t), I(t)) \mid Z(0) = z, I(0) = j],$$

and the derivative $\tilde{\mathscr{L}}$ of $\mathscr{U}_t$ with respect to t at $t = 0$, which is known as the *generator* of the Markov process, by

$$(\tilde{\mathscr{L}} f)(z, j) = \lim_{t \downarrow 0} \frac{(\mathscr{U}_t f)(z, j) - f(z, j)}{t}.$$

Following the standard analysis such as that in [4], the following expression for $\tilde{\mathscr{L}}$ can be derived if $f'(\underline{Z}_i, i) = f'(\bar{Z}_i, i) = 0$:

$$\tilde{\mathscr{L}} f(z, 0) = \tfrac{1}{2}(\sigma_0(z))^2 f''(z, 0) + \mu_0(z) f'(z, 0) + \lambda_0(f(\underline{Z}_1, 1) - f(z, 0)), \quad z \in (\underline{Z}_0, \bar{Z}_0), \tag{1}$$

$$\tilde{\mathscr{L}} f(z, 1) = \tfrac{1}{2}(\sigma_1(z))^2 f''(z, 1) + \mu_1(z) f'(z, 1) + \lambda_1(f(\bar{Z}_0, 0) - f(z, 1)), \quad z \in (\underline{Z}_1, \bar{Z}_1), \tag{2}$$

where $f'(z, i)$ and $f''(z, i)$ are the first and second derivatives of $f(z, i)$ with respect to z.

For $j \in \{0, 1\}$, let

$$F(t, z, i; y, j) = P\{Z(t) \leqslant z, I(t) = i \mid Z(0) = y, I(0) = j\}$$

be the joint distribution of $(Z(t), I(t))$. The analog of the *Kolmogorov forward equation* for F is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i \int_{\underline{Z}_i}^{\bar{Z}_i} f(z, i) F(t, \mathrm{d}z, i; y, j)\right] = \sum_i \int_{\underline{Z}_i}^{\bar{Z}_i} (\tilde{\mathscr{L}} f)(z, i) F(t, \mathrm{d}z, i; y, j) \quad \text{for } y \in (\underline{Z}_j, \bar{Z}_j). \tag{3}$$

Since we assumed that $\sigma_i(z) > 0$ for $z \in [\underline{Z}_i, \bar{Z}_i]$, $F(t, z, i; y, j)$ does not have point masses at any z. Thus we can adopt the density $p(t, z, i; y, j)$ of it. Similarly to the proof of Theorem 1 in [3], using Eqs. (1)–(3) and integrations by parts, we obtain

$$\begin{aligned}
&\sum_i \int_{\underline{Z}_i}^{\bar{Z}_i} f(z, i) \left\{ \frac{1}{2}\sigma_i^2(z) p''(t, z, i; y, j) \right. \\
&\quad + [2\sigma_i(z)\sigma_i'(z) - \mu_i(z)] p'(t, z, i; y, j) - [\mu_i'(z) - (\sigma_i'(z))^2 \\
&\quad \left. -\sigma_i(z)\sigma_i''(z) + \lambda_i] p(t, z, i; y, j) - \frac{\partial}{\partial t} p(t, z, i; y, j) \right\} \mathrm{d}z \\
&\quad + f(\underline{Z}_0, 0) \left\{ \frac{1}{2}(\sigma_0(\underline{Z}_0))^2 p'(t, \underline{Z}_0, 0; y, j) \right. \\
&\quad \left. + [\sigma_0(\underline{Z}_0)\sigma_0'(\underline{Z}_0) - \mu_0(\underline{Z}_0)] p(t, \underline{Z}_0, 0; y, j) \right\} \\
&\quad - f(\bar{Z}_0, 0) \left\{ \frac{1}{2}(\sigma_0(\bar{Z}_0))^2 p'(t, \bar{Z}_0, 0; y, j) \right. \\
&\quad + [\sigma_0(\bar{Z}_0)\sigma_0'(\bar{Z}_0) - \mu_0(\bar{Z}_0)] p(t, \bar{Z}_0, 0; y, j) \\
&\quad \left. -\lambda_1 \int_{\underline{Z}_1}^{\bar{Z}_1} p(t, z, 1; y, j)\,\mathrm{d}z \right\} \\
&\quad + f(\underline{Z}_1, 1) \left\{ \frac{1}{2}(\sigma_1(\underline{Z}_1))^2 p'(t, \underline{Z}_1, 1; y, j) \right. \\
&\quad + [\sigma_1(\underline{Z}_1)\sigma_1'(\underline{Z}_1) - \mu_1(\underline{Z}_1)] p(t, \underline{Z}_1, 1; y, j) \\
&\quad \left. +\lambda_0 \int_{\underline{Z}_0}^{\bar{Z}_0} p(t, z, 0; y, j)\,\mathrm{d}z \right\}
\end{aligned}$$