Operations Research Letters 40 (2012) 492–497

Contents lists available at SciVerse ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Capacitated assortment and price optimization under the multinomial logit model

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ARTICLE INFO

Article history:
Received 27 February 2012
Received in revised form
25 June 2012
Accepted 5 August 2012
Available online 19 August 2012

Keywords:
Multinomial logit model
Assortment optimization
Multi-product price optimization
Efficient assortment

ABSTRACT

We consider an assortment and price optimization problem where a retailer chooses an assortment of competing products and determines their prices to maximize the total expected profit subject to a capacity constraint. Customers' purchase behavior follows the multinomial logit choice model with general utility functions. This paper simplifies it to a problem of finding a unique fixed point of a single-dimensional function and visualizes the assortment optimization process. An efficient algorithm to find the optimal assortment and prices is provided.

1. Introduction and literature review

Many firms face the problem of selecting an assortment of products and determining their prices to maximize the total profit subject to a capacity constraint. For example, a retailer has to choose a product assortment to display given a limited shelf space, and change prices accordingly to meet its business goals. In the joint assortment and price optimization problem, we first have to study customers' purchase behavior and investigate the substitution pattern of features and prices among similar products.

The multinomial logit (MNL) choice model on substitutable products has been an active area of research for several decades. It has received significant attention from researchers of economics, marketing, operations management and transportation science, and has motivated tremendous theoretical research and empirical validations in a large range of applications since it was first proposed and formulated by McFadden [15]. The MNL model is based on a probabilistic model of the individual customer utility and appropriately describes the purchase behavior of customers facing a variety of competing products.

Talluri and van Ryzin [19], Gallego et al. [9] and Liu and van Ryzin [14] study an assortment optimization problem under the MNL model without a capacity constraint and identify the optimal assortment based on ranking products by margins. Chen and Hausman [6] consider the product line and price selection problem and discover the mathematical properties that transfer the combinatorial optimization problem to a convex optimization. Schon [18] further shows that it can be linearized. Rusmevichientong et al. [17] and Wang [20] consider the assortment problem with a capacity constraint and develop a geometric nonrecursive polynomial-time algorithm. Rusmevichientong et al. [16] extend to the nested logit model with dissimilarity factors less than one for consistency with the utility maximization theory, and also develop a polynomial-time approximation scheme. Davis et al. [7] relax the constraint and consider a general model with arbitrary dissimilarity factors. They show that the problem is NP-hard and formulate a tractable convex program whose optimal objective value is an upper bound.

For a multi-product price optimization problem under the MNL model, Hanson and Martin [11] show that the profit function is not concave in prices and they propose a path-following approach to find the globally optimal solution. Other researchers follow different approaches and have observed that the markup, which is price minus cost, is constant across all the products of the firm at the optimal solution (see [1,3,12,10]). The profit function is unimodal and the optimal prices are the unique solution to the first order condition systems (see [2]). Similar results have been observed in the nested logit model (see [13]). Besbes and Saure [5] study the assortment and price competition under the MNL model with a capacity constraint. They point out that each retailer always offers a full-capacity assortment in an equilibrium and the equilibrium assortment has a nested structure: the products can be simply ranked by quality values and costs, and the equilibrium assortment are the top products of full capacity size in the ranking. In all of their models, it is assumed that the utility functions are linear in prices and the price sensitivities are identical for all the

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doi:10.1016/j.orl.2012.08.003

products. Empirical studies have shown that the product-specified price sensitivities may vary widely and the importance of allowing different price sensitivities in the MNL model has been recognized (see [4,8]). Moreover, the utility functions may not be linear in prices for some products in reality. Unfortunately, the markup is no longer constant and the nested structure is lost when price sensitivities are product-differentiated or the utility functions are not linear.

This paper considers capacitated assortment and price optimization under the MNL model with general utility functions. Under very mild conditions, it can be simplified to a problem of finding the fixed point of a decreasing function. For linear utility functions with product-specified price sensitivities, our analysis shows that the adjusted markup, which is price minus cost minus the reciprocal of the price sensitivity, is constant for all the products in an assortment at an optimal solution.

2. The model

Assortment selection and pricing are among the most critical decisions for a firm with variety of products in an increasingly competitive business environment. Attraction models, including the well known MNL model, are the most commonly applied consumer discrete choice models in empirical studies and theoretical research because they capture demand interdependencies and fit actual data well in scenarios where customers choose among a set of alternative products. In this paper, we assume that consumers' purchase behavior follows the MNL model.

The MNL model can be derived from the underlying random utility maximization model. Suppose that $\mathcal{M} := \{1, 2, \ldots, m\}$ is the product set in consideration. The utility U_i of product i at price p_i can be decomposed into two parts: a deterministic component $u_i(p_i)$ and a random component ξ_i, i.e.,

$$U_i = u_i(p_i) + \xi_i.$$

Without loss of generality, the utility of non-purchase option is assumed to be $U_0 = \xi_0$, where ξ_0 is also a random variable. According to the random utility maximization model, the probability that an individual selects alternative i from a given assortment $S \subseteq \mathcal{M}$ with price vector $\mathbf{p}_S := (p_i)_{i\in S}$ is

$$d_i(S, \mathbf{p}_S) = \Pr\Big(U_i \geq \max\big\{U_s : s \in S^+\big\}\Big), \quad \forall i \in S^+, \tag{1}$$

where $S^+ := S \cup \{0\}$.

We further assume that $\{\xi_i, i = 0, 1, \ldots, m\}$ are i.i.d. random variables with a Gumbel or type I generalized extreme value (GEV) distribution:

$$\Pr(\xi_i \leq x) = e^{-e^{-(x+\gamma)}},$$

where γ is Euler's constant ($\gamma \approx 0.5772$). Then, Eq. (1) results in the celebrated MNL model:

$$d_i(S, \mathbf{p}_S) = \frac{e^{u_i(p_i)}}{1 + \sum_{s\in S} e^{u_s(p_s)}}, \quad \forall i \in S. \tag{2}$$

We wish to find an assortment with at most C products and determine their prices to maximize the total expected profit. The unit cost is c_i for product $i \in \mathcal{M}$. The capacitated assortment and price optimization under the MNL model can be formulated as follows

$$\max_{S\subseteq\mathcal{M},\ \mathbf{p}_S} R(S, \mathbf{p}_S) \stackrel{\text{def}}{=} \sum_{i\in S} (p_i - c_i) d_i(S, \mathbf{p}_S), \qquad s.t., |S| \leq C, \tag{3}$$

where $d_i(S, \mathbf{p}_S)$ is defined in Eq. (2) and $|S|$ denotes the cardinality of set S.

2.1. Price optimization

We make some regularity assumptions for the utility functions.

Assumption 1. For each product $i \in \mathcal{M}$, the utility function $u_i(p_i)$ is differentiable and decreasing in price p_i, and $\lim_{p_i\to\infty}(p_i - c_i)e^{u_i(p_i)} = 0$.

Assumption 1 is compatible to the reality: the utility is decreasing in price and that $\lim_{p_i\to\infty}(p_i - c_i)e^{u_i(p_i)} = 0$ is equivalent to $\lim_{p_i\to\infty}(p_i - c_i)d_i(S, \mathbf{p}_S) = 0$ (the so-called null price is infinite here). Let $u_i'(p_i)$ and $u_i''(p_i)$ be the first and second order derivatives of $u_i(p_i)$ in p_i respectively for each $i \in \mathcal{M}$. Under Assumption 1, it is straightforward to verify that

$$\frac{\partial d_i(S, \mathbf{p}_S)}{\partial p_i} = u_i'(p_i) d_i(S, \mathbf{p}_S)(1 - d_i(S, \mathbf{p}_S)) < 0,$$

$$\frac{\partial d_i(S, \mathbf{p}_S)}{\partial p_j} = -u_j'(p_j) d_i(S, \mathbf{p}_S) d_j(S, \mathbf{p}_S) > 0, \quad \forall j \in S,\ j \neq i.$$

The probability of selecting each product is decreasing in its price and increasing in the prices of other products in the assortment. The probability $d_i(S, \mathbf{p}_S)$ is often referred to the market share of product i in a homogeneous market.

Assumption 2. For each product $i \in \mathcal{M}$, the utility function $u_i(p_i)$ is twice-differentiable and concave in p_i.

Assumption 2 is fairly general and is consistent with the risk-averse assumption. Taking an individual's wealth into account, suppose that the deterministic term of the utility function can be expressed as follows: $u_i(p_i) = K_i + V(I - p_i)$, where K_i is the utility from consumption of product i, I is the individual's income level and $V(\cdot)$ is the utility with respect to her net wealth. That $u_i(p_i)$ is decreasing concave is equivalent to that $V(\cdot)$ is increasing concave, i.e., the individual is risk-averse, which is a widely used assumption in psychology, economics and finance. Many utility functions satisfy Assumption 2, e.g., the class of functions $\big\{u_i(p_i) = \alpha_i - \beta_i p_i^{\gamma_i} : \beta_i \geq 0, \gamma_i \geq 1\big\}$.

We first consider the price optimization for each given assortment S. In this step, we only need to consider prices such that the market share is positive for each product in this assortment because we will optimize over all assortments later. The price optimization problem is the following

$$\max_{\mathbf{p}_S} R(S, \mathbf{p}_S), \qquad s.t., d_i(S, \mathbf{p}_S) > 0, \quad \forall i \in S, \tag{4}$$

where $d_i(S, \mathbf{p}_S)$ follows the MNL model (2) and $R(S, \mathbf{p}_S)$ is defined in problem (3).

Proposition 1. *Under Assumption 1, the quantity $p_j^* - c_j + 1/u_j'(p_j^*)$ is constant for all $j \in S$ at a (local or global) optimal price vector $\mathbf{p}_S^*$ to problem (4). Moreover, $p_j^* - c_j + 1/u_j'(p_j^*)$ is also equal to the total profit priced at $\mathbf{p}_S^*$.*

Proof. From the Karush–Kuhn–Tucker (*KKT*) conditions, the necessary conditions are

$$\nabla \sum_{i\in S}(p_i - c_i) d_i(S, \mathbf{p}_S) + \sum_{i\in S} \lambda_i \nabla d_i(S, \mathbf{p}_S) = 0,$$

$$\lambda_i d_i(S, \mathbf{p}_S) = 0, \quad \lambda_i \geq 0,\ \forall i \in S,$$

where λ_i is the Lagrange multiplier. Because $d_i(S, \mathbf{p}_S) > 0$ in problem (4), $\lambda_i = 0$ for any $i \in S$. Then, for each $j \in S$:

$$\begin{aligned}\frac{\partial R(S, \mathbf{p}_S)}{\partial p_j} &= d_j(S, \mathbf{p}_S) + (p_j - c_j) u_j'(p_j) d_j(S, \mathbf{p}_S) \\ &\quad \times \big(1 - d_j(S, \mathbf{p}_S)\big) - \sum_{i\in S, i\neq j} (p_i - c_i) u_j'(p_j) \\ &\quad \times d_i(S, \mathbf{p}_S) d_j(S, \mathbf{p}_S) = 0.\end{aligned}$$