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A modified proximal point algorithm with errors for approximating solution of the general variational inclusion

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ABSTRACT

In this paper, a modified proximal point algorithm with errors, which consists of a resolvent operator technique step with errors followed by a modified orthogonal projection onto a moving half-space, is constructed for approximating the solution of the general variational inclusion in Hilbert space. The convergence of the iterative sequence is shown under weak assumptions. The results improve and extend some known results.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, respectively, and $2^{\mathcal{H}}$ denote the family of all the nonempty subsets of $\mathcal{H}$. Let $M:\mathcal{H}\to 2^{\mathcal{H}}$ be a set-valued mapping, $\mathrm{Graph}(M)=\{(v,u):u\in M(v)\}$ denote the graph of M, and S denote the root set of M, i.e., $S=\{x\in\mathcal{H}:0\in M(x)\}$. Throughout this paper, we assume that $S\neq\emptyset$. We consider the class of general nonlinear variational inclusions: Find $x\in\mathcal{H}$ such that

$$0\in M(x). \tag{1.1}$$

As a matter of fact, problems of minimization or maximization of functions, variational inequality problems, and minimax problems can be unified into the form (1.1) (see [1,11,10,8,4]). This explains why many algorithms have been proposed for its solution, see [11,10,12,13,16,3,2,6,15,5,9,17,14,7]. When M is maximal monotone, Rockafellar [11] introduced the proximal point algorithm, and showed that the sequence $\{x^k\}$, generated from an initial point x^0 by

$$x^{k+1}=J_k(x^k+e^k), \tag{1.2}$$

converges weakly to a solution to (1.1) in R^n, provided the approximation is made sufficiently accurate as the iteration proceeds, where $\{e^k\}$ is an error sequence, $J_k=(I+\lambda_k M)^{-1}$ for a sequence $\{\lambda_k\}$ of positive real numbers that is bounded away from zero.

In 1992, Eckstein and Bertsekas [3] introduced the generalized proximal point algorithm and proved that the sequence $\{x^k\}$, generated from an initial point x^0 by

$$x^{k+1}=(1-\rho_k)x^k+\rho_k w_k,\quad \forall k\geq 0, \tag{1.3}$$

where $\|w_k-J_k(x^k)\|\leq\varepsilon_k$ for sequences $\{\varepsilon_k\}_{k=0}^{\infty}$, $\{\rho_k\}_{k=0}^{\infty}$, $\{\lambda_k\}_{k=0}^{\infty}$ satisfying

$$\sum_{k=0}^{\infty}\varepsilon_k<\infty,\qquad \inf_{k\geq 0}\rho_k>0,\qquad \sup_{k\geq 0}\rho_k<2\quad\text{and}$$
$$\inf_{k\geq 0}\lambda_k>0,$$

converges weakly to a solution to (1.1).

In 2003, based on the projection on the domain of M, He et al. [6] presented a new approximate proximal point algorithm in R^n as follows: for given x^k and $\lambda_k>0$, set

$$x^{k+1}=P_{\Omega}[\bar{x}^k-e^k],\quad \bar{x}^k=J_k(x^k+e^k),$$

where Ω is the domain of M, and $\{e^k\}$ is an error sequence and obeys $\|e^k\|\leq\eta_k\|x^k-\bar{x}^k\|$ with $\sup_{k\geq 0}\eta_k<1$ and $\inf_{k\geq 0}\lambda_k>0$.

In 2005, Yang and He [15], using $\bar{x}^k-x^k$ as the search direction, obtained the inexact iterate $\{x^{k+1}\}$ by

$$\bar{x}^k=J_k(x^k+e^k),\qquad x^{k+1}=P_C(x^k-\rho_k(x^k-\bar{x}^k)),$$

where C is a nonempty closed convex subset of R^n, $\inf_{k\geq 0}\lambda_k>0$, $\|e^k\|\leq\eta_k\|x^k-\bar{x}^k\|$ with $\sum_{k=0}^{\infty}\eta_k^2<+\infty$ and $\{\rho_k\}\subset(0,2)$ is a sequence satisfying $0<\inf_{k\geq 0}\rho_k$, $\sup_{k\geq 0}\rho_k<2$, and proved the convergence of the sequence $\{x^{k+1}\}$.

If the set Ω (or C) is simple enough, so that projections onto it are easily executed, then the methods due to He et al. [6] and Yang

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et al. [15] are useful; but, if Ω (or C) is a general closed and convex set, then a minimal distance problem has to be solved in order to obtain the next iterate. This might seriously affect the efficiency of the approximate proximal point algorithm.

Inspired and motivated by He et al. [6] and Yang et al. [15], in this paper, we replace the projection onto Ω (or C) by a projection onto a specific constructible half-space, and propose a modified algorithm with errors, which consists of a resolvent operator technique step with errors followed by a modified orthogonal projection onto a moving half-space, for approximating the solution of Problem (1.1). We also prove that the iterative sequence $\{x^k\}$ converges weakly to a solution of Problem (1.1) under weak assumptions. Our results improve and extend the corresponding results shown by Rockafellar [11], Eckstein and Bertsekas [3], Yang et al. [15], and Han and He [5].

Suppose that $X \subset \mathcal{H}$ is a nonempty closed convex subset and the distance from z to X is denoted by

$$\mathrm{dist}(z, X) := \inf_{x\in X} \|z - x\|.$$

Let $P_X(z)$ denote the projection of z onto X, that is, $P_X(z)$ satisfies the condition

$$\|z - P_X(z)\| = \mathrm{dist}(z, X).$$

The following well-known properties of the projection operator will be used in this paper. For any $x, y \in \mathcal{H}$ and $z \in X$

(1) $u = P_X(x) \iff \langle u - x, z - u\rangle \ge 0.$
(2) $\|P_X(x) - P_X(y)\| \le \|x - y\|.$
(3) $\|P_X(x) - z\|^2 \le \|x - z\|^2 - \|P_X(x) - x\|^2.$

Definition 1.1. A multi-valued operator M is said to be

(1) monotone if

$$\langle u - v, x - y\rangle \ge 0, \quad \forall x, y \in \mathcal{H},\ u \in M(x),\ v \in M(y);$$

(2) maximal monotone, if M is monotone and $(I + \lambda M)(\mathcal{H}) = \mathcal{H}$ for all $\lambda > 0$, where I denotes the identity mapping on $\mathcal{H}$.

2. Algorithm and convergence

In this section, we shall construct an iterative sequence $\{x^k\}$ for solving Problem (1.1) involving a maximal monotone mapping, and prove its weak convergence.

Algorithm 2.1. *Step* 0. Select an initial $x^0 \in \mathcal{H}$ and set $k = 0$.

Step 1. Find $y^k \in \mathcal{H}$ such that

$$y^k = J_k(x^k + e^k), \tag{2.1}$$

where the positive sequence $\{\lambda_k\}$ satisfies $\alpha := \inf_{k\ge 0} \lambda_k > 0$ and $\{e^k\}$ is an error sequence.

Step 2. Set $K = \{z \in \mathcal{H} : \langle x^k - y^k + e^k, z - y^k\rangle \le 0\}$ and

$$x^{k+1} = (1 - \beta_k)x^k + \beta_k P_K(x^k - \rho_k(x^k - y^k)), \tag{2.2}$$

where $\{\beta_k\}_{n=0}^{+\infty} \subset (0, 1]$ and $\{\rho_k\}_{n=0}^{+\infty} \subset [0, 2)$ are real sequences.

Theorem 2.1. *Let $\{x^k\}$ be the sequence generated by Algorithm 2.1. If*

(i) $\|e^k\| \le \eta_k\|x^k - y^k\|$ *for* $\eta_k \ge 0$ *with* $\sum_{k=0}^{\infty} \eta_k^2 < +\infty$;
(ii) $\{\beta_k\}_{n=0}^{+\infty} \subset [c, d]$ *for some* $c, d \in (0, 1)$;
(iii) $0 < \inf_{k\ge 0} \rho_k$ *and* $\sup_{k\ge 0} \rho_k < 2$;

then the infinite sequence $\{x^k\}$ converges weakly to a solution of Problem (1.1).

Proof. Suppose that $x^* \in \mathcal{H}$ is a solution of Problem (1.1), then we have $0 \in M(x^*)$. We divide the proof of Theorem 2.1 into three steps.

Step 1. We show that $\{x^k\}$ is bounded. From (2.1), it follows that

$$\frac{1}{\lambda_k}(x^k - y^k + e^k) \in M(y^k).$$

By the monotonicity of M, we deduce that

$$\left\langle 0 - \frac{1}{\lambda_k}(x^k - y^k + e^k), x^* - y^k \right\rangle \ge 0,$$

which leads to

$$x^* \in K = \{z \in H : \langle x^k - y^k + e^k, z - y^k\rangle \le 0\}.$$

Let $t^k = P_K(x^k - \rho_k(x^k - y^k))$, we deduce that

$$\langle t^k - (x^k - \rho_k(x^k - y^k)), x^* - t^k\rangle \ge 0,$$

and

$$\begin{aligned}\|x^* - t^k\|^2 &\le \|(x^k - x^*) - \rho_k(x^k - y^k)\|^2 \\ &= \|x^* - x^k\|^2 + \rho_k^2\|x^k - y^k\|^2 \\ &\quad + 2\rho_k\langle x^* - x^k, x^k - y^k\rangle \\ &= \|x^* - x^k\|^2 - \rho_k(2 - \rho_k)\|x^k - y^k\|^2 \\ &\quad + 2\rho_k\langle x^* - y^k, x^k - y^k\rangle \\ &\le \|x^* - x^k\|^2 - \rho_k(2 - \rho_k)\|x^k - y^k\|^2 \\ &\quad + 2\rho_k\langle y^k - x^*, e^k\rangle.\end{aligned}$$

Since $\lim_{k\to\infty} \eta_k = 0$, there exists $k_0 \ge 0$ such that $2\eta_k \le \frac{2-\sup\rho_k}{4} \le \frac{2-\rho_k}{4}$ for all $k \ge k_0$.

From

$$\begin{aligned}2\rho_k\langle y^k - x^*, e^k\rangle &= 2\rho_k\langle y^k - x^k, e^k\rangle + 2\rho_k\langle x^k - x^*, e^k\rangle \\ &\le 2\eta_k\rho_k\|x^k - y^k\|^2 + \frac{4\rho_k\eta_k^2}{2 - \rho_k}\|x^* - x^k\|^2 \\ &\quad + \frac{\rho_k(2 - \rho_k)}{4\eta_k^2}\|e^k\|^2 \\ &\le \left(\frac{\rho_k(2 - \rho_k)}{4} + 2\eta_k\rho_k\right)\|x^k - y^k\|^2 \\ &\quad + \frac{4\rho_k\eta_k^2}{2 - \rho_k}\|x^* - x^k\|^2,\end{aligned}$$

we have that, for all $k \ge k_0$,

$$\begin{aligned}\|x^* - t^k\|^2 &\le \left(1 + \frac{4\rho_k\eta_k^2}{2 - \rho_k}\right)\|x^* - x^k\|^2 \\ &\quad - \frac{\rho_k(2 - \rho_k)}{2}\|x^k - y^k\|^2.\end{aligned} \tag{2.3}$$

Therefore, for all $k \ge k_0$,

$$\begin{aligned}\|x^* - x^{k+1}\|^2 &= \|(1 - \beta_k)(x^* - x^k) + \beta_k(x^* - t^k)\|^2 \\ &= (1 - \beta_k)\|x^* - x^k\|^2 + \beta_k\|x^* - t^k\|^2 \\ &\quad - (1 - \beta_k)\beta_k\|x^k - t^k\|^2 \\ &\le \|x^* - x^k\|^2 + \beta_k\frac{4\rho_k\eta_k^2}{2 - \rho_k}\|x^* - x^k\|^2 \\ &\quad - \beta_k\frac{\rho_k(2 - \rho_k)}{2}\|x^k - y^k\|^2, \\ &\le \|x^* - x^k\|^2 + \frac{4d\rho_k\eta_k^2}{2 - \rho_k}\|x^* - x^k\|^2 \\ &\quad - \frac{c\rho_k(2 - \rho_k)}{2}\|x^k - y^k\|^2.\end{aligned} \tag{2.4}$$