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Separating tight metric inequalities by bilevel programming

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ABSTRACT

We present the first exact approach to separate tight metric inequalities for the Network Loading problem. We give a bilevel programming formulation for the separation problem, develop an algorithm based on the proposed formulation and discuss computational results.

1. Introduction

The Network Loading Problem (NL) can be defined as follows. Given a graph with edge capacity installation costs and a set of traffic demands, choose minimum cost integer capacities such that all the demands can be routed simultaneously on the network. If each demand is restricted to follow a single path, then the flows are said to be unsplittable, otherwise they are called splittable. Let us consider splittable flows, let $G(V, E)$ be an undirected graph and let D be the set of traffic demands (commodities). Without loss of generality, the commodities can be defined as follows [5]: each commodity k has a single source s_k and multiple destinations. For every node i, $d_i^k \geq 0$ is the amount to be sent from s_k to i. Let x_e be an integer variable representing the capacity installed on edge e and let f_{ij}^k and f_{ji}^k be continuous variables representing the flow for commodity k directed from i to j and vice versa on edge $e = (i, j)$. Let $N(i) = \{j \in V : (i, j) \in E\}$ be the neighborhood of node i. The flow formulation of the problem is the following.

$$(FF) \quad \min \sum_{e \in E} c_e x_e$$

$$\sum_{j \in N(i)} (f_{ij}^k - f_{ji}^k) = -d_i^k \quad i \in V, k \in D, i \neq s_k \tag{1}$$

$$\sum_{k \in D} (f_{ij}^k + f_{ji}^k) \leq x_e \quad e = (i, j) \in E \tag{2}$$

$$f, x \geq 0, \quad x \in \mathbb{Z}^{|E|}.$$

Constraints (1) ensure that the demands are satisfied. Constraints (2) guarantee that the installed capacity supports the flow. Since flows are splittable, the problem can also be formulated using only capacity variables. For the formal derivation of the capacity formulation, see for example [3,5].

Definition 1.1. A function $\mu : E \to \mathbb{R}$ is a metric on G if and only if: (i) $\mu_e \geq 0$ for all $e \in E$; (ii) $\mu_e \leq \mu(P_e)$ for all $e \in E$, where $\mu(P_e)$ is the length of shortest path P_e between the endpoints of edge e using μ as edge weights.

Let Met_E be the cone generated by all metrics on G, the capacity formulation of the problem is the following.

$$(CF) \quad \min \sum_{e \in E} c_e x_e$$

$$\sum_{e \in E} \mu_e x_e \geq \sum_{k \in D} \sum_{i \in V} \pi^{\mu}_{s_k i} d_i^k \quad \mu \in \mathrm{Met}_E \tag{3}$$

$$x \in \mathbb{Z}_+^{|E|}.$$

Value $\pi^{\mu}_{s_k i}$ is the length of the shortest path from s_k to i using μ as edge weights. Inequalities (3) are known as *metric inequalities* [15,21]. It is sufficient to consider only integer metrics [5]. For integer metrics the right-hand-side of (3) can be rounded, obtaining the *rounded metric inequalities* (4).

$$\sum_{e \in E} \mu_e x_e \geq \left\lceil \sum_{k \in D} \sum_{i \in V} \pi^{\mu}_{s_k i} d_i^k \right\rceil \quad \mu \in \mathrm{Met}_E \text{ and integer.} \tag{4}$$

Rounded metric inequalities with binary coefficients are known as $\{0, 1\}$-rounded metrics [3]. A special class of $\{0, 1\}$-rounded metrics are the well known cut inequalities, for which conditions to be facet defining are known [17,18]. For undirected graphs, the cut metric is obtained partitioning the nodes into $\{S: V \setminus S\}$ and setting $\mu_{ij} = 1$ if i and j are in different subsets of the partition and zero otherwise; for directed graphs $\mu_{ij} = 1$ if $i \in S, j \in V \setminus S$ and zero otherwise.

Many contributions can be found in the literature for the NL problem and exact and heuristic approaches have been used.

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Heuristic approaches are proposed in [11,12]. In [4] cut inequalities are used to develop a branch-and-cut algorithm. Extending results in [2], cut based inequalities for the flow formulation are used in [23]. A branch-and-bound based on a Lagrangian relaxation is presented in [14]. In [5] the authors compare the flow and the capacity formulation and develop cutting plane algorithms. Results for the flow formulation are given in [13]. Facet defining inequalities are presented in [1]. Capacity formulations based on metric inequalities are used in [7,10,19,20]. In [3] the authors introduce the *tight metric inequalities*, showing that they completely describe the convex hull of integer feasible solutions of the capacity formulation of the problem, and presenting a heuristic approach to separate them. No exact separation algorithm is known in the literature for such inequalities so far.

The aim of this paper is to present the first exact algorithm for separating the tight metric inequalities. The discussion is made for undirected graphs, but the results can be transferred without loss of generality to directed ones. The rest of the paper is structured as follows: in Section 2 the tight metric inequalities and the separation algorithm are presented, in Section 3 implementation details are given, in Section 4 the results obtained using the proposed method on a set of problems derived from real-life instances are discussed.

2. Tight metric inequalities

Let $NL(G, D)$ be the convex hull of integer feasible solutions of the capacity formulation (CF), the following result holds.

Theorem 2.1 *([3]). If $a^T x \geq b$ is valid for $NL(G, D)$, then there exists $\mu \in \text{Met}_E$ such that $\mu^T x \geq b$ is valid and $\mu_e \leq a_e$ for all $e \in E$.*

As a consequence, we can restrict to valid inequalities with metric coefficients. Once μ is chosen, the quality of the corresponding inequality depends on the right-hand-side b. The best possible right-hand-side for a given μ can be computed as $R_\mu = \min\{\mu^T x : x \in NL(G, D)\}$. Inequalities (5) are called *tight metric inequalities* [3].

$$\sum_{e \in E} \mu_e x_e \geq R_\mu \quad \mu \in \text{Met}_E. \tag{5}$$

Since every constraint $\mu^T x \geq b$ is dominated by $\mu^T x \geq R_\mu$, all facet defining inequalities are tight metric inequalities. If μ is an extreme ray of the metric cone, then Theorem 2.2 provides a way to compute R_μ.

Theorem 2.2 *([3]). If μ is an integer valued extreme ray of the metric cone having greatest common divisor equal to one, then $R_\mu = \lceil \sum_{k \in D} \sum_{i \in V} \pi^\mu_{s_k i} d^k_i \rceil$.*

Therefore, rounded metric inequalities defined using extreme rays of the metric cone are tight metrics. Unfortunately, using extreme rays of the metric cone is neither a necessary nor a sufficient condition for obtaining facet defining inequalities. Let G be the complete undirected graph on three nodes and let the demands D be $d^1_2 = d^1_3 = d^2_3 = 1.2$. Inequality $x_{12} + x_{13} + x_{23} \geq 5$ is facet defining, although the corresponding metric μ is not extreme (it can be obtained as the conic combination of the cut metrics induced by partitions $\{\{1\}{:}\{2, 3\}\}$, $\{\{2\}{:}\{1, 3\}\}$, $\{\{3\}{:}\{1, 2\}\}$, with all coefficients equal to 1/2). Let us consider the same graph, but a different set of demands $\bar{D}$ having $\bar{d}^1_2 = 1, \bar{d}^1_3 = \bar{d}^2_3 = 0$. Inequality $x_{13} + x_{23} \geq 0$, corresponding to the metric induced by the cut $\{\{1, 2\}{:}\{3\}\}$, is not facet defining (it is dominated by non-negativity constraints), although the metric is extreme (both sets are connected). Tight metric inequalities are stronger than both metrics and rounded metrics. Consider again G and D as defined above and metric $\mu_{ij} = 1$ for all $i, j \in V$. The right-hand-side of the metric inequality corresponding to μ is 3.6, the one for the rounded metric is 4, but $R_\mu = 5$.

2.1. Separating tight metric inequalities

Given a capacity vector $\bar{x}$, the separation problem for tight metric inequalities can be stated as follows.

Definition 2.3. Find either an inequality $\sum_{e \in E} \mu_e x \geq R_\mu$, $\mu \in \text{Met}_E$ violated by $\bar{x}$ or conclude that none exists.

Since R_μ is the solution of a NL problem having μ as objective coefficients, even computing R_μ when μ is already provided is difficult [3]. Following [16], the separation problem for tight metric inequalities can be formulated as the bilevel programming problem $(sepT)$. Bilevel programming is used to model problems where two players (leader and follower), each of them controlling a subset of the variables, are involved. For details on bilevel programming, see [6,8].

$$(sepT) \quad \min \sum_{e \in E} \bar{x}_e \mu_e - R_\mu$$

$$0 \leq \mu_e \leq U \quad e \in E \tag{6}$$

$$(subsepT) \quad R_\mu = \min \sum_{e \in E} \mu_e y_e$$

$$\sum_{j \in N(i)} (f^k_{ij} - f^k_{ji}) = -d^k_i \quad i \in V, k \in D, i \neq s_k$$

$$\sum_{k \in D} (f^k_{ij} + f^k_{ji}) \leq y_e \quad e = (i, j) \in E$$

$$f \geq 0, \qquad y \in \mathbb{Z}^{|E|}_+.$$

Since $(sepT)$ is a minimization problem, it is not needed, in principle, to require that $\mu \in \text{Met}_E$ (see Theorem 2.1). Constraint (6) is used to prevent unboundedness. Subproblem $(subsepT)$ is the flow formulation of the NL problem for G and D with μ as objective coefficients, that is used to compute R_μ.

Suppose to partition the edges into $\{Y{:}E\setminus Y\}$. If we relax the integrality requirements in $(subsepT)$ for the y variables corresponding to edges not belonging to Y, instead of R_μ a lower bound β on R_μ is computed. Since $\beta \leq R_\mu$, inequalities (7) are valid, and we call them *quasi tight metric inequalities*.

$$\sum_{e \in E} \mu_e x_e \geq \beta. \tag{7}$$

The quality of the bound β (and of the corresponding quasi tight metric) depends on Y. If $Y = E$, quasi tight metrics are tight metrics. If $Y = \emptyset$, quasi tight metrics are metric inequalities. In fact, if no capacity is required to be integer, the optimal capacities for the subproblem can be obtained routing each demand d^k_i on the shortest path from s_k to i using μ as weights and installing on every edge e a capacity equal to the sum of the demands routed on e. Hence $\beta = \sum_{k \in D} \sum_{i \in V} \pi^\mu_{s_k i} d^k_i$ and the bilevel problem can be rewritten as a single level linear programming problem, obtaining the separation problem for metric inequalities.

If μ is integer, quasi tight metric inequalities (7) can be strengthened by rounding β, obtaining the *rounded quasi tight metric inequalities* (8).

$$\sum_{e \in E} \mu_e x_e \geq \lceil \beta \rceil. \tag{8}$$

Rounded quasi tight metric inequalities can be separated solving $(sepR)$.

$$(sepR) \quad \min \sum_{e \in E} \bar{x}_e \mu_e - z$$

$$z < \beta + 1 \tag{9}$$

$$0 \leq \mu_e \leq U \quad e \in E$$

$$z \in \mathbb{Z}, \quad \mu \in \mathbb{Z}^{|E|}$$