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Price of anarchy for supply chains with partial positive externalities

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ABSTRACT

We consider the impact of partial positive externalities (imperfect complementarity) among downstream retailers on supply chain performance. We show that double marginalization may fail to exist in a decentralized setting when some retailers carry multiple imperfect complements. By giving a precise characterization on the degree of complementarity, we prove that a decentralized supply chain loses at least 25% of the optimal profit and that its performance degrades rapidly with the complementarity effect.

1. Introduction

The issue of inefficiency in a decentralized supply chain has attracted a lot of attention since Spengler [22] introduced "double marginalization", i.e., two price markups, imposed by an upstream supplier and downstream retailers. The existing literature typically assumes that the demand of competing retailers exhibits *negative externalities* (or *substitutability*), i.e., an increase in one retailer's price induces an increase in the demand for other retailers' products. Research has shown that substitutability reduces the double marginalization effect, and hence improves the channel performance [2,5].

In this paper, we focus on *positive externalities* (or *complementarity*), i.e., a decrease in the price of one product results in an increase in demand for all other products. We investigate the performance of a supply chain with a single supplier and several downstream retailers. The supplier offers wholesale price contracts to each retailer who carries multiple *imperfect complements*, inducing *partial* positive externalities. Most existing literature on supply chain performance with positive externalities typically assumes *perfect* complements [6,7,25,26], which implies that, whenever a purchase takes place, a consumer has to purchase one product from each and every retailer. Most complements in reality only exhibit partial complementarity – more complement goods are sold compared to the base goods, e.g., games versus game consoles, software versus hardware, ink cartridges versus printers, etc.

Based on a simple example, we illustrate a surprising phenomenon in a multi-product setting with partial complementarity – double marginalization may fail to exist in a decentralized supply chain! By characterizing the degree of complementarity for imperfect complements, we quantify the performance of a decentralized supply chain with respect to the centralized setting with upper and lower bounds. We show that the decentralized supply chain loses *at least* 25% of the optimal profit. We derive two lower bounds on its performance with respect to the complementarity effect, which we will rigorously define later. We present the instances when the bounds are tight and demonstrate their performance in a general setting through numerical simulations.

Discussions on complements in the economics and the industrial organization literature tend to focus on single-tier oligopolistic settings [3,4,12,17]. Most studies on supply chain performance with complements consider assembly chains [6,25–27] as opposed to the distribution channels addressed in this work. [18] studies the impact of supply-side externalities on supply chains, where the complementary effect arises through product availability and prices are exogenous. Our work considers demand-side externalities that arise through prices which are endogenously determined. Our work is also related to a stream of literature on the price of anarchy, which has appeared in transportation networks [8,9,20,21], network pricing [1,14], oligopolistic pricing games in a single tier [10,11], and supply chain games with exogenous pricing [15,16,19].

2. The model

We consider a supply chain with one supplier who offers wholesale price contracts to n retailers who carry a set m of products, where $m \geq n$. Retailer i offers product $\{m_{i-1}+1, \ldots, m_i\}$, where $m_0 := 0$, $m_n := m$, and $m_{i-1} < m_i$ for $i > 1$. When $m = n$, every retailer only carries a single product. In the notation

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below, we adopt a convention in which vectors and matrices appear in boldface. As is traditional in the pricing literature [23,24], we consider affine demand functions, $\mathbf{q}(\mathbf{p}) = \bar{\mathbf{d}} - \mathbf{B}\mathbf{p}$, where $\bar{\mathbf{d}} \geq 0$. We assume that the price sensitivity matrix $\mathbf{B}$ is symmetric, which is a natural consequence of maximizing a quasilinear utility function of a representative consumer. To model the positive externalities or complements, we require that $\partial q_i(\mathbf{p})/\partial p_j \leq 0$ for all i and j, $\partial p_i(\mathbf{q})/\partial q_i < 0$ for all i, and $\partial p_i(\mathbf{q})/\partial q_j \geq 0$ for all $j \neq i$ [24]. The first condition implies that the demand for complements moves in the same direction when the price of one product changes, whereas the other conditions suggest that the prices of complements move in opposite directions when the supply for one product changes. For instance, if the supply for product i increases, i's price decreases. This induces an increase in demand for product i, which then triggers an increase in demand for its complementary product j, resulting in an increase in product j's price. Together with the symmetry assumption on $\mathbf{B}$, this implies that $\mathbf{B}^{-1}$ belongs to the class of M-matrices; the reader is referred to [13] for details. Let $\boldsymbol{\Gamma}$ be a *block* diagonal matrix, consisting of n blocks, whose ith block is the square submatrix of $\mathbf{B}$ formed by the rows and columns indexed $m_{i-1}+1, \ldots, m_i$. $\boldsymbol{\Gamma}$ is referred to as the *intra-firm* price sensitivity matrix. Denote $\bar{\mathbf{B}} = \mathbf{B} - \boldsymbol{\Gamma}$ as the *inter-firm* price sensitivity matrix. For the setting when each retailer carries a single product, $\boldsymbol{\Gamma}$ and $\bar{\mathbf{B}}$ are simplified to the diagonal and off-diagonal matrix of $\mathbf{B}$, respectively.

For each product, we assume that the production capacity is unlimited and that the marginal costs are constant. Let $\mathbf{c}$ denote the vector of marginal costs. Our final assumption is that $\tilde{\mathbf{d}} := \mathbf{B}^{-1}\bar{\mathbf{d}} > \mathbf{c}$, implying that $\bar{\mathbf{d}} > \mathbf{B}\mathbf{c}$, i.e., the base demand at marginal costs must be positive. We will refer to this as Assumption ($\star$).

Throughout the paper, we compare the performance of a decentralized supply chain to a benchmark setting of a centralized setting. Denote the wholesale prices, retail prices, order quantities, and chain-wide profit as $\mathbf{w}$, $\mathbf{p}$, $\mathbf{q}$, and π, respectively. We use superscripts d and c to differentiate the decentralized and the centralized settings.

In a decentralized supply chain, the supplier initiates the process by proposing a wholesale price contract $\mathbf{w_i}$ to every retailer i with the goal of maximizing the profit. Each retailer then determines his/her retail prices $\mathbf{p_i}$, given the prices set by his/her competitors, $\mathbf{p}_{-i}$. We assume that Nash equilibrium has been reached where no single retailer can increase his/her profit by unilaterally changing his/her price. The problem for the supplier and the retailers can be written as follows:

$$\pi_s^d(\mathbf{w}) : \max_{\mathbf{w} \geq \mathbf{0}} (\mathbf{w} - \mathbf{c})^T \mathbf{q}(\mathbf{p}(\mathbf{w})),$$

$$\pi_{r_i}^d(\mathbf{p_i}) : \max_{\mathbf{q_i}(\mathbf{p_i}, \mathbf{p}_{-i}) \geq \mathbf{0}} (\mathbf{p_i} - \mathbf{w_i})^T \mathbf{q_i}(\mathbf{p_i}, \mathbf{p}_{-i}).$$

In a centralized supply chain, a central authority decides production quantities and retail prices across the chain with the objective of maximizing the chain-wide profit by solving the following problem:

$$\pi^c(\mathbf{p}) : \max_{\mathbf{q}(\mathbf{p}) \geq \mathbf{0}} (\mathbf{p} - \mathbf{c})^T \mathbf{q}(\mathbf{p}).$$

Proposition 1. *In a decentralized supply chain,* $\mathbf{w}^d = \frac{1}{2}(\mathbf{B}^{-1}\bar{\mathbf{d}} + \mathbf{c})$, $\mathbf{p}^d = \frac{1}{2}(\mathbf{B} + \boldsymbol{\Gamma})^{-1}(2\mathbf{B} + \boldsymbol{\Gamma})\tilde{\mathbf{d}} + \mathbf{c}$, $\mathbf{q}^d = \frac{1}{2}\mathbf{B}(\mathbf{B} + \boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}\tilde{\mathbf{d}}$, $\pi^d = \frac{1}{4}\tilde{\mathbf{d}}^T(2\mathbf{B} + \boldsymbol{\Gamma})(\mathbf{B} + \boldsymbol{\Gamma})^{-1}\mathbf{B}(\mathbf{B} + \boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}\tilde{\mathbf{d}}$. *In a centralized supply chain,* $\mathbf{p}^c = \frac{1}{2}(\mathbf{B}^{-1}\bar{\mathbf{d}} + \mathbf{c})$, $\mathbf{q}^c = \frac{1}{2}\mathbf{B}\tilde{\mathbf{d}}$, $\pi^c = \frac{1}{4}\tilde{\mathbf{d}}^T\mathbf{B}\tilde{\mathbf{d}}$.

Proof. The decentralized problem is solved by backward induction. Since $\mathbf{B}^{-1}$ is an M-matrix, it is positive definite. This guarantees the existence and uniqueness of a pure strategy equilibrium to the unconstrained problem. We need to show that the solution obtained from the equilibrium condition also satisfies the nonnegativity constraint. The order quantities in the decentralized setting can be written as $\mathbf{q}^d = \frac{1}{2}(\mathbf{B}^{-1} + \boldsymbol{\Gamma}^{-1})^{-1}\tilde{\mathbf{d}}$. Since $\mathbf{B}^{-1}$ is an M-matrix, $\boldsymbol{\Gamma}^{-1} + \mathbf{B}^{-1}$ is also an M-matrix, and its inverse is nonnegative. $\tilde{\mathbf{d}}$ is positive by Assumption ($\star$); thus, we have shown that $\mathbf{q}^d \geq 0$. Similarly, we can show that $\mathbf{w}^d$ satisfies the nonnegativity constraint. For the centralized problem, $\mathbf{q}^c$, which is a product of a nonnegative matrix and a positive vector, is clearly nonnegative. □

3. Comparative studies on prices and quantities

In this section, we begin with one example which highlights a interesting phenomenon that occurs in a decentralized supply chain with partial positive externalities. Consider the following setting with two retailers and three products. Retailer 1 carries the first two products and retailer 2 carries the third product. The price sensitivity matrix, marginal cost, and the maximum demand are given as follows:

$$\mathbf{B} = \begin{bmatrix} 0.657 & 0.231 & 0.284 \\ 0.231 & 0.422 & 0.154 \\ 0.284 & 0.154 & 0.611 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 0.878 \\ 0.290 \\ 0.979 \end{bmatrix},$$

$$\bar{\mathbf{d}} = \begin{bmatrix} 1.432 \\ 1.373 \\ 1.267 \end{bmatrix}.$$

In a decentralized supply chain, the wholesale prices and the retail prices are given by $\mathbf{w}^d = (0.879, 1.336, 1.025)$ and $\mathbf{p}^d = (0.871, 1.856, 1.116)$, respectively. Notice that $p_1^d - w_1^d = -0.008$. Retailer 1 is unable to cover the wholesale price, and loses money every time product 1 is sold!

This example illustrates a characteristic unique to pricing of complements: the base product (product 1 in the example) is priced low to generate a sufficient sales volume which stimulates the demand for its complements (product 2). The objective is to create a level of profit which adequately covers losses sustained by the base product (otherwise, the retailer is better off by exiting the market). The almost universal tactic in the desktop printer business involves printers selling for as little as \$100, including two ink cartridges, which themselves cost around \$30 each to replace. Thus the company prices low on the printers to create the anticipated revenue flow from selling the ink cartridges.

Proposition 2. *In a supply chain with partial positive externalities,*
(a) $\mathbf{w}^d = \mathbf{p}^c > \mathbf{c}$, *and*
(b) $\mathbf{q}^d \leq \mathbf{q}^c$.

Proof. (a) The inequality can be easily established under Assumption ($\star$). (b) The order quantities in the decentralized setting could be expressed as $\mathbf{q}^d = \frac{1}{2}(\mathbf{B}^{-1} + \boldsymbol{\Gamma}^{-1})^{-1}\tilde{\mathbf{d}}$. Using the inverse binomial theorem, we can express the term $(\mathbf{B}^{-1} + \boldsymbol{\Gamma}^{-1})^{-1}$ as $\mathbf{B} - (\mathbf{B}^{-1}\boldsymbol{\Gamma}^{1/2}\boldsymbol{\Gamma}^{1/2}\mathbf{B}^{-1} + \mathbf{B}^{-1})^{-1}$. $\mathbf{B}^{-1}\boldsymbol{\Gamma}^{1/2}$, $\boldsymbol{\Gamma}^{1/2}\mathbf{B}^{-1}$, and $\mathbf{B}^{-1}$ are M-matrices; thus the second term is nonnegative. It follows that $(\mathbf{B}^{-1} + \boldsymbol{\Gamma}^{-1})^{-1} \leq \mathbf{B}$. Since $\tilde{\mathbf{d}}$ is positive, we obtain the desired result. □

The supplier in the decentralized setting and the central planner in the centralized setting charge the same prices and keep a positive markup for every product they distribute. Any product whose price falls below the manufacturing cost will be dropped. The example above illustrates that, for certain products, the retail price in a decentralized setting could be lower than in the centralized setting. Nonetheless, Proposition 2 states that, for every product, fewer units are sold in the decentralized setting.

4. Comparative studies on the channel profit

Decentralized supply chains are widely recognized as less efficient than centralized settings. The ratio π^d/π^c compares the