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A constructive characterization of the split closure of a mixed integer linear program

Juan Pablo Vielma

School of Industrial and Systems Engineering, Georgia Institute of Technology, 765 Ferst Drive, Atlanta, GA 30332-0205, USA

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Abstract

Two independent proofs of the polyhedrality of the split closure of mixed integer linear program have been previously presented. Unfortunately neither of these proofs is constructive. In this paper, we present a constructive version of this proof. We also show that split cuts dominate a family of inequalities introduced by Köppe and Weismantel.

Keywords: Mixed-integer linear programming; Lattice bases; Cutting planes; Split cuts; Split closure

1. Introduction

In 1990 Cook et al. [8] introduced a family of cuts for a mixed integer linear program (MILP) which they called split cuts. These cuts are a special case of Balas' disjunctive cuts [4] which arise from a particular two term disjunction. Split cuts are also related to intersection cuts introduced by Balas in 1971 [3]. A precise correspondence between split cuts and intersection cuts has been established for 0–1 MILPs by Balas and Perregaard [5] and for general MILPs by Andersen et al. [1,2].

The split closure of a MILP is the convex set defined by the intersection of all of its split cuts. Cook et al. [8] proved that the split closure of a MILP is a polyhedron. Andersen et al. [1,2] have given an alternate proof of this fact. Unfortunately, neither of these proofs is constructive in the sense that they do not provide a method for constructing the split closure for a given MILP.

Another family of cutting planes related to split cuts is the one introduced by Köppe and Weismantel in 2004 [9]. This family of cuts is based on a mixed integer version of the Farkas' Lemma and they were related to split cuts by Bertsimas and Weismantel in 2005 [6].

By using an algebraic characterization of split cuts introduced by Caprara and Letchford [7] we are able to show that every cut from [9] is dominated by the split cut to which it is related. Furthermore, by using this relationship and a result from [1,2] we are able to construct a finite set of split cuts that define the

E-mail address: jvielma@isye.gatech.edu.

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split closure, hence providing a constructive proof of its polyhedrality. The key step of this proof is using the characterization from [7] to note that every non-dominated split cut for a particular relaxation of a MILP can be associated to an element in a lattice introduced in [9].

The rest of the paper is organized as follows. In Section 2 we introduce some notation, the algebraic characterization of split cuts from [7] and some results from [1,2] we will use later. Then, in Section 3 we present a simplified characterization of split cuts for a particular relaxation of the MILP. Finally, in Section 4 we use this simplified characterization show that the cutting planes introduced in [9] are dominated by split cuts and develop the constructive proof of the polyhedrality of the split closure.

2. Split cuts

We study the feasible region of a MILP problem given by

$$P_I := \{x \in P \subseteq \mathbb{R}^n : x_j \in \mathbb{Z}\ \forall j \in N_I\},$$

where $N = \{1, \dots, n\}$, $N_I \subseteq N$ and P is a rational polyhedron given by

$$P := \{x \in \mathbb{R}^n : Ax \leqslant b\},$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $M = \{1, \dots, m\}$, $r = \operatorname{rank}(A)$ and $a_{i\cdot}$ corresponds to row i of A. We will assume that $P \neq \emptyset$, but we will not assume $r = n$ allowing for P to contain a line. We also allow for P to be not full dimensional.

Now, let

$$\mathscr{B}_r^* := \{B \subseteq M : |B| = r \text{ and } \{a_{i\cdot}\}_{i\in B} \text{ are linearly independent}\}.$$

Then, for every $B \in \mathscr{B}_r^*$ we define the following relaxation of P:

$$P(B) := \{x \in \mathbb{R}^n : a_{i\cdot}^{\mathrm{T}} x \leqslant b_i\ \forall i \in B\}.$$

Note that $\mathscr{B}_n^*$ corresponds to the bases of P, so for simplicity we will refer to $B \in \mathscr{B}_r^*$ as a basis even when $r < n$, noting that in this later case, feasible bases will not define extreme points of P. In any case we will define $\bar{x}(B)$ to be a particular, but arbitrarily selected, solution to $a_{i\cdot}^{\mathrm{T}} x = b_i$, $\forall i \in B$.

We will study *split disjunctions* $D(\pi, \pi_0)$ of the form $\pi^{\mathrm{T}} x \leqslant \pi_0 \vee \pi^{\mathrm{T}} x \geqslant \pi_0 + 1$ where $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$. We denote the set of points satisfying split disjunction $D(\pi, \pi_0)$ as

$$F_{D(\pi,\pi_0)} := \{x \in \mathbb{R}^n : \pi^{\mathrm{T}} x \leqslant \pi_0 \vee \pi^{\mathrm{T}} x \geqslant \pi_0 + 1\}$$

and $\operatorname{conv}(P \cap F_{D(\pi,\pi_0)})$ as the *disjunctive hull* defined by P and $D(\pi, \pi_0)$. Similarly for $B \in \mathscr{B}_r^*$ we define the *basic disjunctive hull* defined by B and $D(\pi, \pi_0)$ as $\operatorname{conv}(P(B) \cap F_{D(\pi,\pi_0)})$.

We say that a disjunction $D(\pi, \pi_0)$ is valid for P_I if $P_I \subseteq F_{D(\pi,\pi_0)} \subsetneq \mathbb{R}^n$. We are interested in the following set of disjunctions, which are always valid for P_I.

$$\Pi_0^n(N_I) := \{(\pi, \pi_0) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} : \pi_j = 0,\ j \notin N_I\}.$$

We also define the projection of $\Pi_0^n(N_I)$ into the π variables as

$$\Pi^n(N_I) := \{\pi \in \mathbb{Z}^n \setminus \{0\} : \pi_j = 0,\ j \notin N_I\}.$$

With this, the *split closure* of P_I is defined as

$$SC := \bigcap_{(\pi,\pi_0)\in \Pi_0^n(N_I)} \operatorname{conv}(P \cap F_{D(\pi,\pi_0)}).$$

Similarly, for $B \in \mathscr{B}_k^*$ we define the *basic split closure* as

$$SC(B) := \bigcap_{(\pi,\pi_0)\in \Pi_0^n(N_I)} \operatorname{conv}(P(B) \cap F_{D(\pi,\pi_0)}).$$

A *split cut* is an inequality valid for SC and hence valid for P_I. Similarly a *basic split cut* is an inequality valid for $SC(B)$ for some $B \in \mathscr{B}_r^*$. It is known that basic split cuts are exactly the same as intersection cuts (see, for example, [1,2]).

If $\delta^{\mathrm{T}} x \leqslant \delta_0$ and $\gamma^{\mathrm{T}} x \leqslant \gamma_0$ are two inequalities valid for SC, we will say that $\delta^{\mathrm{T}} x \leqslant \delta_0$ is *dominated* by $\gamma^{\mathrm{T}} x \leqslant \gamma_0$ if and only if

$$\{x \in P : \gamma^{\mathrm{T}} x \leqslant \gamma_0\} \subseteq \{x \in P : \delta^{\mathrm{T}} x \leqslant \delta_0\}.$$

Similarly, if the inequalities are valid for $SC(B)$ for some $B \in \mathscr{B}_r^*$, we will say that $\delta^{\mathrm{T}} x \leqslant \delta_0$ is *dominated* by $\gamma^{\mathrm{T}} x \leqslant \gamma_0$ if and only if

$$\{x \in P(B) : \gamma^{\mathrm{T}} x \leqslant \gamma_0\} \subseteq \{x \in P(B) : \delta^{\mathrm{T}} x \leqslant \delta_0\}.$$

In particular, we will say that a split cut or basic split cut $\delta^{\mathrm{T}} x \leqslant \delta_0$ is *non-trivial* if and only if it is not