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Exact solution to a Lindley-type equation on a bounded support

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Abstract

We derive the limiting waiting-time distribution F_W of a model described by the Lindley-type equation $W = \max\{0, B - A - W\}$, where B has a polynomial distribution. This exact solution is applied to derive approximations of F_W when B is generally distributed on a finite support. We provide error bounds for these approximations.

Keywords: Alternating service; Contraction mapping; Linear differential equation; Polynomial distribution

1. Introduction

Consider a server alternating between two service points. At each service point there is an infinite queue of customers waiting to be served. Only one customer can occupy each service point. Once a customer enters the service point, his total service is divided into two separate phases. First there is a preparation phase, where the server is not involved at all. After the preparation phase is completed the customer is allowed to start with the second phase, which is the actual service. The customer either has to wait for the server to return from the other service point, where he may be still busy with the previous customer, or he may commence with his actual service immediately after completing his preparation phase. This would be the case only if the server had completed serving the previous customer and was waiting for this customer to complete his preparation phase. The server is obliged to alternate; therefore, he serves all odd-numbered customers at one service point and all even-numbered customers at the other. Once the service is completed, a new customer immediately enters the empty service point and starts his preparation phase without any delay. In the above setting, the steady-state waiting time of the server W is given by the Lindley-type equation (see also [10])

$$W = \max\{0, B - A - W\}, \tag{1}$$

where B and A are the steady-state preparation and service time respectively.

It is interesting to note that this equation is very similar to Lindley's equation. The only difference

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between the two equations is the sign of W at the right-hand side. Lindley's equation describes the relation between the waiting time of a customer W and the interarrival time A and service time B in a single server queue. It is one of the fundamental and most well-studied equations in queuing theory. For a detailed study of Lindley's equation we refer to [1,4] and the references therein.

The model described by (1) applies in many real-life situations that involve a single server alternating between two stations. It was first introduced in [7], which studies a two-carousel bi-directional system that is operated by a single picker. In this setting, the preparation time B represents the rotation time of the carousels and A is the time needed to pick an item. It is assumed that B is uniformly distributed, while the pick time A is either exponential or deterministic. The authors are mainly interested in the steady-state waiting time of the picker. This problem is further investigated in [11], where the authors expand the results in [7] by allowing the pick times to follow a phase-type distribution.

In general, it is not possible to derive a closed-form expression for the distribution of W for every given distribution F_B of B (or F_A of A). In [10] the authors derive an exact solution under the assumption that A is generally distributed and B follows a phase-type distribution. For the classic Lindley-equation, the M/G/1 single server queue is perhaps the easiest case to analyse. The analogous scenario for our model would be to allow the service time A to be exponentially distributed and the preparation time B to follow a general distribution. For this model though, the analysis is not straightforward, as is the case for Lindley's equation. The structure of F_B (or the lack thereof) is essential for this model. If F_B belongs to a specific class of distributions, exact computations are possible. This class of distributions includes at least all distribution functions that have a rational Laplace transform and a density on an unbounded support. Both this class and the closed-form expression for the distribution of W are described in detail in [9].

Despite the fact that this class is fairly big, it does not include all distribution functions. For example, if F_B is a Pareto distribution, the method described in [9] is inapplicable. Polynomial distributions are another example of distributions that do not belong to this class. However, they are extremely useful, since they can be used to approximate any distribution function that has a bounded support. Our main goal in this paper is to complement the above mentioned results by deriving a closed-form expression of the steady-state distribution of the waiting time, F_W, under the assumption that A is exponentially distributed and B follows a polynomial distribution. In Section 2 we derive F_W under these assumptions. As an application, in Section 3 we discuss how one can use this result in order to derive good approximate solutions for F_W when B is generally distributed on a bounded support, and we provide error bounds of these approximations. We conclude in Section 4 with some numerical results.

2. Exact solution of the waiting-time distribution

In this section we derive a closed-form expression of F_W, under the assumption that A is exponentially distributed and B follows a polynomial distribution. Without loss of generality we can assume that F_B has all its mass on $[0, 1]$. Therefore, let

$$F_A(x) = 1 - \mathrm{e}^{-\mu x} \quad \text{and}$$

$$F_B(x) = \begin{cases} \sum_{i=0}^{n} c_i x^i & \text{for } 0 \leqslant x \leqslant 1, \\ 1 & \text{for } x \geqslant 1, \end{cases} \tag{2}$$

where $\sum_{i=0}^{n} c_i = 1$. Let $X = B - A$. As we have shown in [9, Section 4], the mapping

$$(\mathcal{T}F)(x) = 1 - \int_x^{\infty} F(y - x)\, \mathrm{d}F_X(y) \tag{3}$$

is a contraction mapping—with the contraction constant equal to $\mathbb{P}[B > A]$—in the space $\mathcal{L}^{\infty}([0, \infty))$, i.e., the space of measurable and bounded functions on the real line with the norm

$$\|F\| = \sup_{x \geqslant 0} |F(x)|.$$

Furthermore, we have shown that F_W, provided that F_A or F_B is continuous, is the unique solution to the fixed-point equation $F = \mathcal{T}F$. Then from (3), for