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# Counting the number of renewals during a random interval in a discrete-time delayed renewal process

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**Abstract**

In a discrete-time delayed renewal process, we study the distribution of the number of renewals during a random interval. We obtain closed-form expressions for the probability mass function and binomial moments of this number for various distributions of the random interval and interrenewal times.


*Keywords:* Delayed renewal process; Counting process; Discrete-time

## 1. Introduction

This paper is concerned with counting the number of renewals in a discrete-time random interval. Let $N(n)$ be the number of renewals in a fixed discrete-time interval $(0, n]$, where $n$ is a positive integer. The interrenewal times occur according to a sequence of discrete random variables $\{X_1, X_2, \ldots, X_i, \ldots\}$, where $X_1$ is started at time 0. Let $T$ be a random variable representing a discrete-time interval, which is independent of $\{X_1, X_2, \ldots, X_i, \ldots\}$. Hence $N(T)$ is a random variable which represents the number of renewals occurring in the random interval $(0, T]$. For the sake of convenience, we call $T$ a session time in this paper.

The problem of finding the probability distribution of $N(T)$ in the continuous-time setting has been treated for several specific cases by Cox in his monograph [3, p. 42] under the title "the number of renewals in a random time." Most of the results presented by Cox are based on the *ordinary* renewal process, i.e., all the random variables $X_i, i = 1, 2, \ldots$ come from the same distribution [3, p. 25]. Thus it is more general to consider the case in which $X_i, i = 2, 3, \ldots$ come from the same distribution as $X_2$ while only $X_1$ may come from a different distribution. Such a case is called the *delayed* renewal process [3, p. 28]. As a special case of the delayed renewal process, if $X_1$ is a residual life of $X_2$, we have the *equilibrium* renewal process [3, p. 28]. These are the three types of renewal processes introduced by Cox, and often considered by others subsequently. However, as a further generalization of the delayed renewal process, we may assume that each

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interrenewal time $X_1, X_2, \ldots$ can have a different distribution. The authors have extended Cox's treatment of counting the number of renewals in a random time interval to these general delayed renewal processes, with special application to the calculation of the number of handovers during a conversational session in cellular mobile communication networks [7,8].

The discrete-time renewal process is dealt with in several textbooks, including [4,5], where it is called the *recurrent event process*. They present a theory for counting the number of renewals in a fixed time interval. As an application, Koutras [6] discusses the number of appearances of a specific pattern in a fixed number of repeated trials. More applications of this type can be found in [1]. However, none of these studies has considered the number of renewals in a random time interval. Such occasions arise in many cases. For example, in a digital communication system, a session of transmitting a long message such as file downloading from a web site may be segmented into several packets based on the Internet Protocol. In this case, the number of packets in a message can be counted by using the present model. Since each packet needs its own header and trailer, it is necessary to know the number of packets generated from this session in order to calculate the total transmission time. A classic example of handling a special pattern in the bit sequence is the *bit stuffing* in the High-level Data Link Control (HDLC) protocol [2, p. 88]. A 0 is stuffed after each consecutive five 1's within a frame in order to avoid confusion with the flag, 01111110, indicating the end of the frame. Our model can be used to find the number of stuffed 0's in a frame of random length.

## 2. General session time and general interrenewal times in a delayed renewal process

In Sections 2–4, we assume a delayed renewal process, i.e., a sequence of interrenewal times $\{X_i; i = 1, 2, \ldots\}$ such that $X_i$, $i=2, 3, \ldots$ come from the same distribution as $X_2$. In this section, we present a framework for handling the case in which the session time $T$ and the interrenewal times $X_1$ and $X_2$ have general distributions, respectively.

Let us define the sum of $m$ interrenewal times $\{X_1, X_2, \ldots, X_m\}$ as

$$S_m := \sum_{i=1}^{m} X_i, \quad m = 1, 2, \ldots$$

and let $S_0 :=0$. Let $N(n)$ be a random variable representing the number of renewals in a *fixed* interval $(0, n]$, where $n$ is a positive integer. Thus, the event $\{N(n) \geqslant m\}$ is equivalent to the event $\{S_m \leqslant n\}$, i.e., the number of renewals by time $n$ inclusive is not fewer than $m$ if and only if the $m$th renewal occurs before or at time $n$. Thus we have

$$\begin{aligned} P[N(n) = m] &= P[N(n) \geqslant m] - P[N(n) \geqslant m + 1] \\ &= P[S_m \leqslant n] - P[S_{m+1} \leqslant n]. \end{aligned}$$

Hence

$$P[N(n) = m] = F_{S_m}(n) - F_{S_{m+1}}(n), \quad m = 0, 1, 2, \ldots, \tag{1}$$

where $F_{S_m}(n) := P[S_m \leqslant n]$ is the cumulative distribution function (cdf) of the random variable $S_m$. Note that $F_{S_0}(n) \equiv 1$. We define the probability generating function (pgf) for $N(n)$ as

$$G_{N(T)}(n, z) := \sum_{m=0}^{\infty} P[N(n) = m] z^m, \quad |z| \leqslant 1. \tag{2}$$

Substituting (1) into (2), we get

$$G_{N(T)}(n, z) = 1 + (z - 1) \sum_{m=1}^{\infty} F_{S_m}(n) z^{m-1}. \tag{3}$$

Now, let us define the following generating function:

$$G^*_{N(T)}(y, z) := \sum_{n=1}^{\infty} G_{N(T)}(n, z) y^n, \quad |y| < 1. \tag{4}$$

From (3) and (4) we have

$$G^*_{N(T)}(y, z) = \frac{y}{1 - y} + (z - 1) \sum_{m=1}^{\infty} z^{m-1} \times \sum_{n=1}^{\infty} F_{S_m}(n) y^n. \tag{5}$$