Operations Research Letters 39 (2011) 457–460

Contents lists available at SciVerse ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Integer-empty polytopes in the 0/1-cube with maximal Gomory–Chvátal rank

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ARTICLE INFO

Article history:
Received 6 March 2011
Accepted 13 September 2011
Available online 25 September 2011

Keywords:
Integer programming
Cutting planes
Gomory–Chvátal closure
Rank

ABSTRACT

We provide a complete characterization of all polytopes $P \subseteq [0,1]^n$ with empty integer hulls, whose Gomory–Chvátal rank is n (and, therefore, maximal). In particular, we show that the first Gomory–Chvátal closure of all these polytopes is identical.

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1. Introduction

The Gomory–Chvátal procedure is a well-known technique to derive valid inequalities for the integral hull P_I of a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$. It was introduced by Chvátal [2] and, implicitly, by Gomory [6–8] as a means to establish certain combinatorial properties via cutting-plane proofs. Cutting planes and Gomory–Chvátal cuts, in particular, belong to today's standard toolbox in integer programming. However, despite significant progress in recent years (see, e.g., [1,3,5,9]), the Gomory–Chvátal procedure is still not fully understood from a theoretical standpoint, especially in the context of polytopes contained in the 0/1-cube. For example, the question whether the currently best known upper bound of $O(n^2 \log n)$ on the Gomory–Chvátal rank, established in [5], is tight, remains open. In [5], it was also shown that there is a class of polytopes contained in the n-dimensional 0/1-cube whose rank exceeds n. (See [11] for a more explicit construction.) However, no family of polytopes in the 0/1-cube is known that realizes super-linear rank, and thus there is a large gap between the best known upper bound and the largest realized rank.

We consider the special case of $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and Gomory–Chvátal rank $\mathrm{rk}(P) = n$ (i.e., maximal rank, as $\mathrm{rk}(P) \le n$ holds for all $P \subseteq [0,1]^n$ with $P_I = \emptyset$; see [1]). This case is of particular interest as, so far, all known proofs of polynomial upper bounds on the rank of polytopes in the 0/1-cube (cf., [1,5]) crucially depend on this special case. The improvement from $O(n^3 \log n)$ in [1] to $O(n^2 \log n)$ in [5] as an upper bound on the rank of polytopes in $[0,1]^n$ is a direct consequence of a better upper bound on the rank of certain polytopes in the 0/1-cube that do not contain integral points. It can actually be shown that lower bounds on the rank of polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ play a crucial role in understanding the rank of *any* (well-defined) cutting-plane procedure [10]. Moreover, in many cases, the rank of a face $F \subseteq P$ with $F_I = \emptyset$ induces a lower bound on the rank of P itself. In fact, the construction of the aforementioned families of polytopes in $[0,1]^n$ whose rank is strictly larger than n exploits this connection.

In view of this, a thorough understanding of the Gomory–Chvátal rank of polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ might help to derive better upper and lower bounds for the general case. In this paper, we characterize all polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and $\mathrm{rk}(P) = n$. In particular, we show that after applying the Gomory–Chvátal procedure once, one always obtains the same polytope. Furthermore, we show that $P \subseteq [0,1]^n$ with $P_I = \emptyset$ has $\mathrm{rk}(P) = n$ if and only if $P \cap F \neq \emptyset$ for all one-dimensional faces F of the 0/1-cube $[0,1]^n$.

The paper is organized as follows. In Section 2, we introduce our notation and recall some basic facts about the Gomory–Chvátal procedure. Afterwards, in Section 3, we derive the characterization of all polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and $\mathrm{rk}(P) = n$. In particular, in Section 3.2, we relate the rank of a polytope $P \subseteq [0,1]^n$ with $P_I = \emptyset$ to the rank of its faces. We then prove the characterization for the two-dimensional case in Section 3.3, which is an essential ingredient for the subsequent generalization to arbitrary dimension in Section 3.4.

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doi:10.1016/j.orl.2011.09.004

2. Preliminaries

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polytope with $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$. The *Gomory–Chvátal closure* of P is defined as

$$P' := \bigcap_{\lambda\in\mathbb{R}^m_+, \lambda A\in\mathbb{Z}^n} \{x : \lambda Ax \leq \lfloor \lambda b \rfloor\}.$$

The result P' is again a polytope (see [2]), and one can apply the operator iteratively. We let $P^{(i+1)} := (P^{(i)})'$ for $i \geq 0$ and $P^{(0)} := P$. The resulting sequence $\{P^{(i)}\}_{i\geq 0}$ becomes stationary after finitely many steps [2], and the smallest k such that $P^{(k+1)} = P^{(k)}$ is the *Gomory–Chvátal rank of* P (in the following often *rank of* P), denoted by $\mathrm{rk}(P)$. In particular, $P^{(\mathrm{rk}(P))} = P_I$, where $P_I := \mathrm{conv}(P \cap \mathbb{Z}^n)$ denotes the *integral hull of* P.

We will make repeated use of the following well-known lemma:

Lemma 2.1 (*[4, Lemma 6.33]*). *Let P be a rational polytope and let F be a face of P. Then $F' = P' \cap F$.*

If $P \subseteq [0, 1]^n$ and $P_I = \emptyset$, Lemma 2.1 can be used to derive an upper bound on $\mathrm{rk}(P)$.

Lemma 2.2 (*[1, Lemma 3]*). *Let $P \subseteq [0, 1]^n$ be a polytope with $P_I = \emptyset$. Then $\mathrm{rk}(P) \leq n$.*

This bound is actually tight; a family of polytopes $A_n \subseteq [0, 1]^n$ with $(A_n)_I = \emptyset$ and $\mathrm{rk}(A_n) = n$ was described in [3, p. 481].

For $i \in [n]$, the i-th coordinate flip maps $x_i \mapsto 1-x_i$ and $x_j \mapsto x_j$ for $i \neq j$. Another property that we will extensively use is that the Gomory–Chvátal operator is commutative with unimodular transformations, in particular, coordinate flips.

Lemma 2.3 (*[5, Lemma 4.3]*). *Let $P \subseteq [0, 1]^n$ be a polytope and let u be a coordinate flip. Then $(u(P))' = u(P')$.*

Given polytopes $P \subseteq [0, 1]^n$, $Q \subseteq [0, 1]^k$, and a k-dimensional face F of $[0, 1]^n$, we say that $P \cap F \cong Q$ if the canonical projection of $P \cap F$ onto $[0, 1]^k$ is equal to Q. We denote the *interior of* P by $\mathrm{Int}(P)$ and, with P, F, and Q as before, the *relative interior of* P *with respect to* F is defined as $\mathrm{RInt}_F(P) := \mathrm{Int}(Q)$. We use e to denote the all-one vector, and $\frac{1}{2}e$ to denote the all-one-half vector. If $I \subseteq [n] \times \{0, 1\}$, $\frac{1}{2}e^I$ has coordinates $\frac{1}{2}e^I_i = \frac{1}{2}$ whenever $(i, l) \notin I$, and $\frac{1}{2}e^I_i = l$ for $(i, l) \in I$. Similarly, if F is a face of $[0, 1]^n$, we define $\frac{1}{2}e^F \in F$ to be $\frac{1}{2}$ in those coordinates not fixed by F. Moreover, we define F_k to be the set of all vectors $x \in \left\{0, \frac{1}{2}, 1\right\}^n$ such that exactly k coordinates are equal to $\frac{1}{2}$, and the remaining coordinates are in $\{0, 1\}$. For convenience, we use $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

3. Polytopes $P \subseteq [0, 1]^n$ with $P_I = \emptyset$ and maximal rank

For $n \in \mathbb{N}$, we define the polytope $B_n \subseteq [0, 1]^n$ by

$$B_n := \left\{ x \in [0, 1]^n \,\middle|\, \sum_{i\in S} x_i + \sum_{i\in[n]\setminus S} (1 - x_i) \geq 1 \text{ for all } S \subseteq [n] \right\}.$$

Note that, $(B_n)_I = \emptyset$. This family of polytopes will be essential to our subsequent discussion.

3.1. Properties of B_n

In the following section, we will characterize $B_n^{(k)}$ and show, specifically, that $B_n^{(n-2)} = \left\{\frac{1}{2}e\right\}$. Moreover, we will show that $\left\{0, \frac{1}{2}\right\}$-cuts, i.e., Gomory–Chvátal cuts with $\lambda \in \left\{0, \frac{1}{2}\right\}^m$, suffice to deduce $(B_n)_I = \emptyset$, and the rank with respect to the classical Gomory–Chvátal procedure coincides with the rank if one were to use $\left\{0, \frac{1}{2}\right\}$-cuts only. Clearly, with B_n as above and F being a k-dimensional face of $[0, 1]^n$, we have $B_n \cap F \cong B_k$. As a direct consequence of the proof of [3, Lemma 7.2] one obtains:

Lemma 3.1. *Let $P \subseteq [0, 1]^n$ be a polytope with $F_k \subseteq P$ for some $k < n$. Then $F_{k+1} \subseteq P'$.*

Proof. We include a proof for completeness. Let P as above and let $ax < b + 1$ with $a \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$ be valid for P. We have to show that $ap \leq b$ for every $p \in F_{k+1}$. Let $p \in F_{k+1}$ be arbitrary. If $ap \in \mathbb{Z}$, we are done. So assume that $ap \notin \mathbb{Z}$. Then there exists $i \in [n]$ such that $a_i \neq 0$ and $p_i = \frac{1}{2}$. We define the points p^0, p^1 by setting $p^0_j = p^1_j = p_j$ for all $j \neq i$, $p^0_i = 0$, and $p^1_i = 1$. Hence, $p = \frac{1}{2}p^0 + \frac{1}{2}p^1$. Note that, $p^0, p^1 \in F_k \subseteq P$ and, therefore, $ap^l < b + 1$ holds for $l \in \{0, 1\}$. We derive $ap + \frac{1}{2} \leq \max\{ap^0, ap^1\} < b + 1$ and thus $ap < b + \frac{1}{2}$. Since $ap \in \frac{1}{2}\mathbb{Z}$, it follows that $ap \leq b$, hence $p \in P'$. As the choice of $p \in F_{k+1}$ was arbitrary, we obtain $F_{k+1} \subseteq P'$. □

Note that, $F_2 \subseteq B_n$. Thus, by Lemma 3.1, we have:

Corollary 3.2. *$F_k \subseteq B_n^{(k-2)}$.*

The following theorem specifies a family of valid inequalities for $B_n^{(k)}$.

Theorem 3.3. *Let B_n be defined as above and $k \leq n$. Then*

$$\sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I} (1 - x_i) \geq 1$$

is valid for $B_n^{(k)}$ for all $I \subseteq \tilde{I} \subseteq [n]$ with $|\tilde{I}| = n - k$. Moreover, these inequalities can be derived as iterated $\left\{0, \frac{1}{2}\right\}$-cuts.

Proof. The proof is by induction on k. First, let us look at the case $k = 0$. By definition, $\sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I}(1 - x_i) \geq 1$ with $\tilde{I} = [n]$ is valid for B_n. Now consider $0 < k \leq n$, and assume that the claim holds for $k - 1$. Let $\tilde{I} \subseteq [n]$ with $|\tilde{I}| = n - k$ be arbitrary. We have to prove that $\sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I}(1 - x_i) \geq 1$ with $I \subseteq \tilde{I}$ is valid for $B_n^{(k)}$. Let $I_0 = \tilde{I} \cup \{h\}$ for some $h \notin \tilde{I}$. Note that, such an h exists as $k > 0$. Then

$$x_h + \sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I}(1 - x_i) = \sum_{i\in I\cup\{h\}} x_i + \sum_{i\in I_0\setminus(I\cup\{h\})}(1 - x_i) \geq 1$$

and

$$(1 - x_h) + \sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I}(1 - x_i) = \sum_{i\in I} x_i + \sum_{i\in I_0\setminus I}(1 - x_i) \geq 1$$

are valid for $B_n^{(k-1)}$, by induction hypothesis. By adding the two inequalities, we obtain

$$2\sum_{i\in I} x_i + 2\sum_{i\in\tilde{I}\setminus I}(1 - x_i) \geq 1$$

and, therefore, $\sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I}(1 - x_i) \geq \left\lceil \frac{1}{2} \right\rceil = 1$ is valid for $B_n^{(k)}$. □

We immediately obtain the following corollary.

Corollary 3.4. $B_n^{(n-2)} = \left\{\frac{1}{2}e\right\}$.

Proof. First note that, $\frac{1}{2}e \in B_n^{(n-2)}$ by Corollary 3.2. By Theorem 3.3 we know that $\sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I}(1 - x_i) \geq 1$ with $I \subseteq \tilde{I} = \{u, v\} \subseteq I$ is valid for $B_n^{(n-2)}$, for any pair $u, v \in [n]$, $u \neq v$. Therefore $x_u + x_v \geq 1$, $x_u + (1 - x_v) \geq 1$, $(1 - x_u) + x_v \geq 1$, and $(1 - x_u) + (1 - x_v) \geq 1$ are valid for $B_n^{(n-2)}$, which implies $x_u = x_v = \frac{1}{2}$. □

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