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On the k edge-disjoint 2-hop-constrained paths polytope

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Abstract

The k edge-disjoint 2-hop-constrained paths problem consists in finding a minimum cost subgraph such that between two given nodes s and t there exist at least k edge-disjoint paths of at most 2 edges. We give an integer programming formulation for this problem and characterize the associated polytope.

Keywords: Survivable network; Edge-disjoint paths; Hop-constraint; Polyhedron; Facet

1. Introduction

Given a graph $G = (N, E)$ with $s, t \in N$, a 2-*st*-path in G is a path between s and t of length at most 2, where the length of a path is the number of its edges (also called *hops*). Given a function $c : E \to \mathbb{R}$ which associates a cost $c(e)$ to each edge $e \in E$, the *k edge-disjoint 2-hop-constrained paths problem* (kHPP) is to find a minimum cost subgraph such that between s and t there exist at least k edge-disjoint 2-*st*-paths.

In this paper, we give an integer programming formulation for the kHPP and discuss its associated polytope. In particular, we give a minimal complete linear description of that polytope.

The kHPP arises within the framework of survivable network design problems. Indeed, basic requirements, like the 2-edge connectivity for example, are often not sufficient to guarantee an effective survivable network. In fact, for some types of networks (like ATM and IP networks), a higher level of connectivity is required. Also hop-constrained paths are needed to assure the quality of the (re)routing.

Moreover, the kHPP can be seen as a special case of the more general problem when more than one

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pair of terminals is considered. This is the case, for instance, when several commodities have to be routed in the network. Thus an efficient algorithm for solving the kHPP would be useful to solve (or produce upper bounds for) this more general problem.

Despite these interesting applications, we do not have any knowledge of a previous study of the kHPP. Huygens et al. [6] have already investigated the case where $k=2$ and the bound L on the length of the paths is 2 or 3. They present a complete and minimal linear description of its associated polytope. There has been however a considerable amount of research on many related problems. In [3], Dahl and Johannessen consider the 2-path network design problem which consists of finding a minimum cost subgraph connecting each pair of terminal nodes by at least one path of length at most 2. In [1], Dahl considers the hop-constrained path problem, that is the problem of finding between two distinguished nodes s and t a minimum cost path with no more than L edges when L is fixed. He gives a complete description of the dominant of the associated polytope when $L \leqslant 3$. Thus this hop-constrained path problem corresponds to the special case $k=1$ in kHPP. A main idea in the completeness proof in [1] for the case $L=2$ turns out to be applicable to the case of a general $k \geqslant 1$ (when $L=2$). This is the basis for our completeness result stated in Theorem 6 in Section 3. Dahl and Gouveia [2] consider the directed hop-constrained path problem. They describe valid inequalities and characterize the associated polytope when $L \leqslant 3$.

Given a graph $G=(N, E)$ and an edge subset $F \subseteq E$, the $0-1$ vector $x^F \in \mathbb{R}^E$, such that $x^F(e)=1$ if $e \in F$ and $x^F(e)=0$ otherwise, is called the *incidence vector* of F. The convex hull of the incidence vectors of the solutions to the kHPP on G, denoted by $P_k(G)$, will be called the kHPP *polytope*. Given a vector $w \in \mathbb{R}^E$ and an edge subset $F \subseteq E$, we let $w(F)=\sum_{e\in F} w(e)$. For two node subsets $W_1, W_2 \subset N$, we note $[W_1, W_2]$ the set of edges having one node in W_1 and the other in W_2. If $W_1=\{w_1\}$, we will write $[w_1, W_2]$ for $[\{w_1\}, W_2]$. If $W \subset N$ is a node subset of G, we denote $N \backslash W$ by $\overline{W}$. The set of edges that have only one node in W is called a *cut* and denoted by $\delta(W)$. We will write $\delta(v)$ for $\delta(\{v\})$. A cut $\delta(W)$ such that $s \in W$ and $t \in \overline{W}$ will be called an *st-cut*.

If x^F is the incidence vector of the edge set F of a solution to the kHPP, then clearly x^F satisfies the following inequalities:

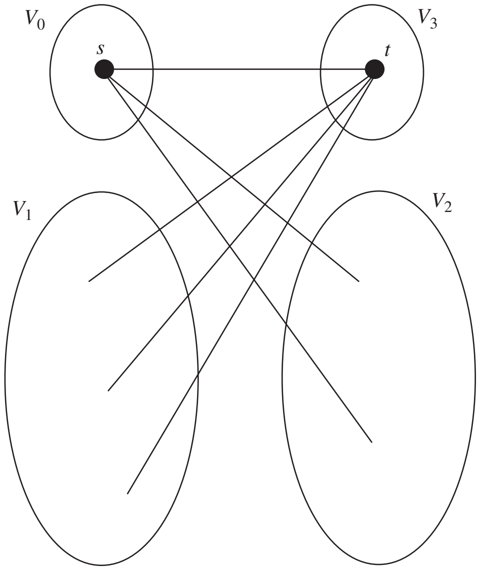

Fig. 1. Support graph of a 2-path-cut inequality.

$$x(\delta(W)) \geqslant k \quad \text{for all } st\text{-cut } \delta(W), \tag{1}$$

$$1 \geqslant x(e) \geqslant 0 \quad \text{for all } e \in E. \tag{2}$$

Inequalities (1) will be called *st-cut inequalities* and inequalities (2) *trivial inequalities*.

In [1], Dahl introduces a class of inequalities valid for the dominant of the hop-constrained path problem. For the special case of $L=2$, they are as follows.

Let V_0, V_1, V_2, V_3 be a partition of N such that $s \in V_0$, $t \in V_3$ and $V_i \neq \emptyset$ for $i=1,2$. Let T be the set of edges $e=uv$ where $u \in V_i$, $v \in V_j$ and $|i-j|>1$. Then the inequality

$$x(T) \geqslant 1$$

is valid for the 2-path polyhedron.

Using the same partition, this inequality can be generalized in a straightforward way to the kHPP polytope as

$$x(T) \geqslant k. \tag{3}$$

The set T is called a 2-*path-cut* and a constraint of type (3) is called a 2-*path-cut inequality*. See Fig. 1 for an example of a 2-path-cut inequality with $V_0=\{s\}$ and $V_3=\{t\}$.