Operations Research Letters 37 (2009) 16–20

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Computational experience with general cutting planes for the Set Covering problem

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ARTICLE INFO

Article history:
Received 9 September 2007
Accepted 30 September 2008
Available online 17 October 2008

Keywords:
Set Covering
Cutting planes
Exact separation

ABSTRACT

In this paper we present a cutting plane algorithm for the Set Covering problem. Cutting planes are generated by running an "exact" separation algorithm over the subproblems defined by suitably small subsets of the formulation constraints. Computational results on difficult small-medium size instances are reported.

1. Introduction

Let $E = \{e_1, \dots, e_m\}$ be a finite set and let $S = \{S_1, \dots, S_n\}$ be a given collection of subsets of E. Let $F \subset \{1, \dots, n\}$ be an index subset. F is said to cover E if $E = \bigcup_{j \in F} S_j$.

The *Set Covering Problem* (SCP) is to find a minimum weighted cover of E. SCP is generally NP-hard, and has relevant applications in Crew Scheduling, Vehicle Routing, Machine Learning.

Let $\boldsymbol{c} = (c_1, c_2, \dots, c_n)$ be a set of weights associated with the elements of E. Let $\boldsymbol{A} = (a_{ij})$ be a matrix with entries $a_{ij} \in \{0, 1\}$, where $a_{ij} = 1$ if $e_i \in S_j$, 0 otherwise, and let $\boldsymbol{1}$ denote a vector of ones of appropriate size. *SCP* can be formulated as:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{1} \\ & \boldsymbol{x} \in \{0, 1\}^{|E|}. \end{aligned}$$

Let $\gamma(\boldsymbol{A}) = \{\boldsymbol{x} \in \{0, 1\}^{|E|} : \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{1}\}$ denote the set of the feasible solutions of *SCP*. We denote by $P(\boldsymbol{A}) = conv(\gamma(\boldsymbol{A}))$ the *Set Covering polytope*. All the nontrivial facets of $P(\boldsymbol{A})$ are of the form $\alpha^T \boldsymbol{x} \geq \beta$, with $\alpha, \beta \geq 0$ [18].

Most of the literature on Set Covering algorithms focused on heuristics for large-scale instances [6–8].

Much less attention has been paid to the exact solution of difficult instances. The only recent approaches we are aware of are the Mannino and Sassano [16] enumerative algorithm for the Steiner Triples and the disjunctive cutting plane algorithm proposed by Ferris, Pataki and Schmieta [11] to solve to optimality the well-known "*seymour*" instance.

The structure of the Set Covering polytope has been deeply investigated in Balas and Ng [4,5], Cornuejols and Sassano [9], Sassano [18], Nobili and Sassano [17], Saxena [19–21], but these relevant theoretical results have not led yet to a successful cutting plane algorithm. In our opinion this is due to the difficulty of designing efficient separation algorithms.

In this paper we report on a computational experience with a separation procedure for general (i.e. not based on the "template paradigm") cutting planes – *SepGcuts* – based on the following idea:

(i) identify a suitably small subproblem defined by a subset of the formulation constraints;
(ii) run an exact separation algorithm over the subproblem to produce a violated cutting plane, if any exists.

The approach can be seen as an early attempt to extend to other IP problems the "local cuts" methodology introduced by Applegate et al. [1,2] for the TSP. This topic has been recently investigated by D. Espinoza in his Ph.D. dissertation [10].

SepGcuts combines MIP separation of rank-1 Chvátal–Gomory cuts to find a "separating subproblem", whose investigation ensures to return a violated valid inequality, and a brute-force separation routine for "suitably small" subproblems, to produce violated facets of the Set Covering polytope $P(\boldsymbol{A})$.

The remainder of the paper is organized as follows. In Section 2 we outline *SepGcuts*. In Section 3 we describe the exact separation procedure for a subproblem $P(\boldsymbol{A}_S)$ of $P(\boldsymbol{A})$ with a reduced number

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0167-6377/$ – see front matter © 2008 Published by Elsevier B.V.
doi:10.1016/j.orl.2008.09.009

of columns and a reduced number of rows. In Section 4 we report on a computational experience on difficult small-medium size instances. In the Appendix we give the implementation details of the MIP separation procedure for rank-1 Chvátal–Gomory cuts.

2. Outline of the separation procedure *SepGcuts*

Let N denote the index set of the columns of $\boldsymbol{A}$. Given a subset $S \subset N$ of columns, in the following we indicate with $\boldsymbol{A}_S$ the submatrix of $\boldsymbol{A}$ defined by the columns of $\boldsymbol{A}$ with index in S and by the rows of $\boldsymbol{A}$ containing only variables in S.

SepGcuts consists of three basic steps. First we look for a submatrix $\boldsymbol{A}_S$ of $\boldsymbol{A}$ with the property that $\hat{\boldsymbol{x}}_S \notin P(\boldsymbol{A}_S)$, where $\hat{\boldsymbol{x}}$ is the current fractional solution and $\hat{\boldsymbol{x}}_S = \left[\hat{\boldsymbol{x}}_i : i \in S\right]$ its support vector.

Then we run an exact separation procedure to identify facet-inducing inequalities of $P(\boldsymbol{A}_S)$ which cut-off $\hat{\boldsymbol{x}}_S$. By "exact separation" we mean a brute-force separation algorithm which guarantees to return a hyperplane separating $\hat{\boldsymbol{x}}_S$ and $P(\boldsymbol{A}_S)$.

In the third step, sequential lifting is used to convert the facets of $P(\boldsymbol{A}_S)$ into facets of the Set Covering polytope $P(\boldsymbol{A})$.

For the success of the procedure it is crucial to choose a set S such that $\hat{\boldsymbol{x}} \notin P(\boldsymbol{A}_S)$ and S is suitably small. In this paper we test rank-1 Chvátal–Gomory cutting planes as drivers to "promising" S subsets, because they are suitably "sparse", so leading to small S subsets. The separation procedure *SepGcuts* can be summarized as follows:

Procedure *SepGcuts*

1. **Choose $\boldsymbol{A}_S$**

1a. Let $\boldsymbol{Bx} \geq \boldsymbol{b}$ the current formulation of the Set Covering problem, after the addition of some valid inequalities. We recall that both $\boldsymbol{B}$ and $\boldsymbol{b}$ have nonnegative entries. Run an MIP separation routine for rank-1 Chvátal–Gomory cuts [12,14] to generate a valid inequality $\boldsymbol{c}^T\boldsymbol{x} \geq d$ for $P(\boldsymbol{A})$ which cuts-off $\hat{\boldsymbol{x}}$, i.e. $\boldsymbol{c}^T\hat{\boldsymbol{x}} < d$

1b. Let $S = \{j \in N : c_j > 0\}$ be the *support of* $\boldsymbol{c}$ and let $\boldsymbol{c}_S = [c_j : j \in S]$ be the *support vector of* $\boldsymbol{c}$.

1c. Set $x_j = 1$ for each $j \in N \setminus S$. Let $\boldsymbol{A}_S$ be the submatrix of $\boldsymbol{A}$ defined by the columns of $\boldsymbol{A}$ with index in S and by the rows of $\boldsymbol{A}$ containing only variables in S (all the inequalities in $\boldsymbol{Ax} \geq \boldsymbol{1}$ including variables in $N \setminus S$ become redundant and can be removed).

2. **Exact separation over** $P(\boldsymbol{A}_S)$

2a. Let $\boldsymbol{x}_S = [x_i : i \in S]$ be the support vector of $\boldsymbol{x}$ and let $\gamma(\boldsymbol{A}_S) = \{\boldsymbol{x}_S \in \{0,1\}^S : \boldsymbol{A}_S\boldsymbol{x}_S \geq \boldsymbol{1}\}$ be the set of the feasible solutions of the reduced Set Covering problem defined by the submatrix $\boldsymbol{A}_S$. Let $P(\boldsymbol{A}_S) = conv(\gamma(\boldsymbol{A}_S))$ be the Set Covering polytope associated with $\boldsymbol{A}_S$.

2b. The inequality $\boldsymbol{c}_S^T\boldsymbol{x}_S \geq d$ is valid for $P(\boldsymbol{A}_S)$ and cuts-off $\hat{\boldsymbol{x}}_S$.

2c. Run an exact separation procedure over $P(\boldsymbol{A}_S)$ to generate an inequality of the form $\alpha_S^T\boldsymbol{x}_S \geq \beta$, $(\alpha_S, \beta) \geq 0$, facet-inducing for $P(\boldsymbol{A}_S)$ and violated by $\hat{\boldsymbol{x}}$.

3. **Lifting**

3a. Any valid inequality for $P(\boldsymbol{A}_S)$ is also valid for $P(\boldsymbol{A})$ [18]. But in general it is not true that the facets of $P(\boldsymbol{A}_S)$ are facet-defining for $P(\boldsymbol{A})$ too, and sequential lifting is required to convert $\alpha_S^T\boldsymbol{x}_S \geq \beta$ into a facet of $P(\boldsymbol{A})$.

3. Exact separation of valid inequalities for $P(\boldsymbol{A}_S)$

Let $P(\boldsymbol{A}_S)$ be the Set Covering polytope associated with the reduced matrix $\boldsymbol{A}_S$ and let $ext(P(\boldsymbol{A}_S))$ denote the set of the extreme points of $P(\boldsymbol{A}_S)$. The linear program for the exact separation of valid inequalities for $P(\boldsymbol{A}_S)$ is:

$$\theta^* = \min\ \hat{\boldsymbol{x}}_S^T\alpha_S - \beta$$
$$\boldsymbol{y}^T\alpha_S \geq \beta, \quad \boldsymbol{y} \in ext(P(\boldsymbol{A}_S)) \tag{1}$$
$$(\alpha, \beta) \in \Omega \tag{2}$$

where Ω is a *normalization set* ensuring that the LP (1)–(2) is not unbounded (and consequently that at least an extreme optimal solution exists).

The following proposition – which can be extended to any 0-1 Integer Programming problem – shows that if Ω is defined by a *normalization hyperplane*, i.e. $\Omega = \{(\alpha_S, \beta) \in \mathbb{R}^{|S|+1} : \pi\alpha_S - \gamma\beta = \pi_0\}$, the extreme points of (1)–(2) are in one-to-one correspondence with the facets of $P(\boldsymbol{A}_S)$.

Proposition 1. *Let $(\overline{\alpha}_S, \overline{\beta})$ be an extreme point solution of the LP:*

$$\theta^* = \min\ \hat{\boldsymbol{x}}_S^T\alpha_S - \beta$$
$$\boldsymbol{y}^T\alpha_S \geq \beta, \quad \boldsymbol{y} \in ext(P(\boldsymbol{A}_S)) \tag{3}$$
$$\pi\alpha_S - \gamma\beta = \pi_0 \tag{4}$$

where $\pi\alpha_S - \gamma\beta = \pi_0$ is a normalization hyperplane. The inequality $\overline{\alpha}_S^T\boldsymbol{x}_S \geq \overline{\beta}$ induces a facet of $P(\boldsymbol{A}_S)$.

Proof. Let us assume that $\boldsymbol{A}_S\boldsymbol{x} \geq \boldsymbol{1}$ has at least two entries for each row, so $P(\boldsymbol{A}_S)$ is full-dimensional. Since $(\overline{\alpha}_S, \overline{\beta})$ is an extreme point solution, it provides $|S|$ linearly independent active constraints (3), i.e. $|S|$ linearly independent feasible solutions $\boldsymbol{y}$ satisfying $\overline{\alpha}_S^T\boldsymbol{y} \geq \overline{\beta}$ as an equality (roots). ◇

It follows that if $\Omega = \{(\alpha_S, \beta) \in \mathbb{R}^{|S|+1} : \pi\alpha_S - \gamma\beta = \pi_0\}$ is a normalization set, the LP (3)–(4) returns a facet of $P(\boldsymbol{A}_S)$ as an extreme point optimal solution.

Now we show that the hyperplane $\{(\alpha_S, \beta) \in \mathbb{R}^{|S|+1} : \boldsymbol{1}^T\alpha_S - \beta = 1\}$ is a normalization hyperplane.

Proposition 2. *The equality $\boldsymbol{1}^T\alpha_S - \beta = 1$ defines a normalization hyperplane.*

Proof. To show that equality (2) is a normalization hyperplane, we project out the variable $\beta = \boldsymbol{1}^T\alpha_S - 1$ to get the equivalent LP:

$$\theta^* = \max\ (\boldsymbol{1} - \hat{\boldsymbol{x}}_S)^T\alpha_S - 1$$
$$(\boldsymbol{1} - \boldsymbol{y})^T\alpha_S \leq 1, \quad \boldsymbol{y} \in ext(P(\boldsymbol{A}_S)). \tag{5}$$

We can assume wlog that, for each $h \in S$, the solution $y_h = 0$ and $y_j = 1$ for each $j \in S \setminus \{h\}$ is feasible and from (5) we get $\alpha_j \leq 1$, for each $j \in S$. It follows that LP (5) is not unbounded, since all the objective function coefficients are nonnegative. ◇

Remark 1. Here we show that the more popular $\beta = 1$ condition does not define a normalization hyperplane. Consider the Set Covering problem:

$$\begin{array}{ll} \min & y1 + 3y2 + 3y3 + 3y4 \\ & y1 + y2 \geq 1 \\ & y1 + y3 \geq 1 \\ & y3 + y4 \geq 1 \\ & y1, y2, y3, y4 \in \{0, 1\} \end{array}$$