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A one-to-one correspondence between colorings and stable sets

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ABSTRACT

Given a graph G, we construct an auxiliary graph $\tilde{G}$ with $\overline{m}$ vertices such that the set of all stable sets of $\tilde{G}$ is in **one-to-one** correspondence with the set of all colorings of G. Then, we show that the Max-Coloring problem in G reduces to the Maximum Weighted Stable set problem in $\tilde{G}$.

1. Introduction

Given a simple graph G, with vertex-set $V(G)$ and edge-set $E(G)$, a *stable set* of G is a subset of vertices any two of which are nonadjacent, and a *coloring* of G is a partition of $V(G)$ into nonempty stable sets. The maximum cardinality of a stable set of G is denoted by $\alpha(G)$, and the minimum number of stable sets in a coloring of G, that is the chromatic number of G, is denoted by $\chi(G)$.

The (classical) *graph coloring problem* is to find a coloring with a minimum number of stable sets. This problem arises in many applications (*e.g.*, timetabling, bandwith allocation...). However, extra constraints or a modified objective function are necessary to model properly most applications.

For instance, given a graph G with a weight $p(v) \in \mathbb{R}$ associated to each vertex v, the *Max-Coloring* problem is to find a coloring $\{V_1, \ldots, V_k\}$ of G that minimizes $\sum_{i=1}^{i=k} \max_{v \in V_i} p(v)$. It has crucial applications in batch scheduling [4,5], buffer minimization [11] and telecommunication [12]. Notice that Max-Coloring is a proper generalization of the classical graph coloring problem, since if p is the unit vector, $\sum_{i=1}^{i=k} \max_{v \in V_i} p(v) = k$. Max-Coloring is substantially harder than the graph coloring problem (for instance it is NP-hard in chordal graphs [4] while graph coloring is almost trivial in these graphs).

Several mathematical programs have been designed in order to solve the coloring problem and its generalizations [1,3,10]. Our approach is very similar to the model of "canonical representatives" proposed in [1] (resp. [9]), where graph coloring (resp. Max-Coloring) is formulated with a 0-1 linear program with $n + \overline{m}$ variables (as usual n is the number of vertices of G and $\overline{m} = O(n^2)$ is the number of edges in the complementary graph of G). We make fundamental improvement on [1,3,9] by formulating the graph coloring problem with a 0-1 linear program (with only $\overline{m}$ variables) of a *well-studied type*: We reduce it to the maximum stable set problem in an auxiliary graph (that is, the problem of determining a stable set with maximum cardinality). The benefit is that any method or bound developed to cope with maximum stable sets, for instance those in [7], can be used directly to cope with minimum colorings (via the reduction that we discuss). Notice that there exists already one well-known reduction from the coloring problem in G to the maximum cardinality stable set problem in an auxiliary graph $\hat{G}$ with order $O(n^2)$: Recall that $\alpha(\hat{G}) + \chi(G) = 2n$, where $\hat{G}$ consists in n disjoint copies $G_1, \ldots, G_n$ of G associated with n new vertices $x_1, \ldots, x_n$ such that each vertex x_i is linked to all vertices of G_i and such that all copies of a vertex of G are pairwise linked in $\hat{G}$. This reduction is actually at the base of compact formulations such as [3]. However, we stress two points. First there are, in general, an exponential number of stable sets of $\hat{G}$ associated with one coloring of G. Second, given a weight function on the vertices of G, we are not aware of a possibility of finding a weight function on the vertices of $\hat{G}$ such that an optimal coloring for Max-Coloring in G corresponds to a stable set of $\hat{G}$ with maximum weight. The reduction that we propose seems to be the first with the following two properties:

- It describes a **1-to-1** mapping between all colorings of G and all stable sets of $\tilde{G}$;
- Max-Coloring reduces to the weighted version of the maximum stable set problem in $\tilde{G}$.

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The paper is organized as follows. In Section 2, we explain how to construct a graph $\tilde{G}$ on $\overline{m}$ vertices whose stable sets are in 1-to-1 mapping with the colorings of a graph G. In Section 3, we show how to generalize our reduction to the Max-Coloring problem. In Section 4 we discuss issues concerning our reduction such as complexity, polyhedra and bounds.

2. A one-to-one correspondence

A *partition into cliques* of G is a partition $\{Q_1, \ldots, Q_k\}$ of $V(G)$ such that each Q_i induces a complete subgraph. Notice that $\{Q_1, \ldots, Q_k\}$ is a partition into cliques of G if and only if it is a coloring of the complementary graph $\overline{G}$. Our reduction is easier to explain in terms of partition into cliques, rather than in terms of colorings. In Section 2.1, we mention how to represent a partition into cliques of a graph by a forest of the same graph. In Section 2.2, we break the symmetries of such a representation, leading to the reduction itself, which is detailed in Section 2.3.

2.1. Forests inducing cliques

The first idea in our reduction relies on the well-known following fact:

Lemma 2.1. *For any graph G and any forest $F \subseteq E(G)$, if $\{C_1, \ldots, C_k\}$ are the connected components of the partial subgraph $(V(G), F)$, we have: $|F| + k = |V(G)|$.* □

In other words, minimizing the number of connected components induced by a forest is equivalent to maximizing the number of edges of a forest. Moreover, finding the minimum number of cliques required to partition $V(G)$ is equivalent to find a forest $F \subseteq E(G)$ with $|F|$ as big as possible such that each tree of F spans a clique. However, representing partitions into cliques with forests suffers two drawbacks. First, a given clique might be spanned by many different trees. Second, the (hereditary) hypergraph of forests inducing cliques requires to be studied as a new combinatorial structure.

In the next section, we put restrictions on the trees that we consider so that each clique of G is spanned by a unique tree. This restriction uses an (arbitrary) acyclic orientation of the edges of the graph. We therefore deal with necessary concepts in oriented graphs.

2.2. Simplicial stellar forests in digraphs

Let D be a simple digraph with vertex-set $V(D)$ and arc-set $A(D)$. An arc with *tail* u and *head* v is denoted by (u, v). A subset of arcs $S \subseteq A(D)$, is called an *out-star* of D if there exists a vertex u such that each arc of S is of the form (u, v) for some $v \in V(D)$. The out-star S is said to be *centered* on the vertex u. The nonempty out-star S *spans* the set $U := \{u : (u, v) \in S\} \cup \{v : (u, v) \in S\}$ of all vertices incident to an arc of S. Two stars are said to be *vertex-disjoint* if they span disjoint subsets of vertices. The digraph D is called *acyclic* if it has no directed cycle. Recall that D is acyclic if and only if there is a total ordering $<$ on its vertex-set such that $u < v$ for each arc (u, v).

A pair of arcs $\{a, b\}$ of D is called a *simplicial pair of D* if $a = (u, v)$, $b = (u, w)$, and (v, w) or (w, v) is an arc of D, for three distinct vertices u, v, w. If all pairs of arcs of an out-star S are simplicial, then S is a *simplicial star of D*. A *simplicial stellar forest of D* is the union of pairwise vertex-disjoint simplicial stars.

Lemma 2.2 explains the relevance of simpliciality with respect to partition into cliques.

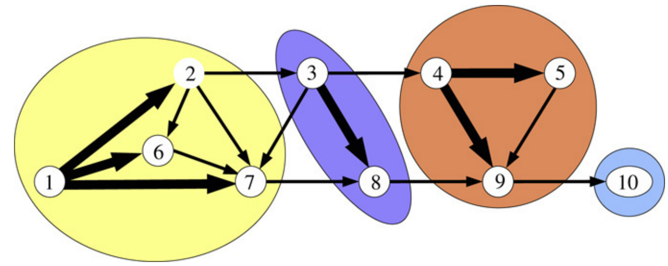

Fig. 1. The correspondence between a partition into cliques and a simplicial stellar forest.

Lemma 2.2. *For any acyclic digraph D, the partitions into cliques of D are in one-to-one mapping with the simplicial stellar forests of D.*

Proof. Each simplicial star spans a unique clique. So we can associate to each simplicial stellar forest $\mathcal{S}$ a unique set of disjoint cliques. These cliques, together with singletons (each vertex not spanned by S is taken as a singleton) form a unique partition into cliques of D. Conversely, since D is acyclic, any clique C, with $|C| \geq 2$, is spanned by a unique simplicial star (centered on the minimum vertex of C assuming $V(G) = \{1, \ldots, n\}$ and $i < j$ for each arc (i, j)). Hence, given a partition into cliques $\mathcal{C} = \{C_1, C_2 \ldots C_k\}$, there exists a unique simplicial stellar forest $\mathcal{S} = \{S_1, S_2 \ldots S_t\}$ such that each clique C_i of $\mathcal{C}$ with $|C_i| \geq 2$ is spanned by a star S_j of $\mathcal{S}$. □

Fig. 1 illustrates the correspondence of Lemma 2.2 (observe that cliques of size 1 in a partition of $V(G)$ correspond to no arc at all in the simplicial stellar forest). Now, and thanks to Lemma 2.3 below, the simplicial stellar forests of a digraph D correspond to the stable sets of an auxiliary graph with $|A(D)|$ vertices. In the next section, we define this auxiliary graph in more details.

Lemma 2.3. *For any digraph D, a subset $A' \subseteq A(D)$ of arcs forms a simplicial stellar forest of D if and only if any two adjacent arcs in A' form a simplicial pair of D.*

Proof. Since necessity is straightforward we only show sufficiency. Assume that any two adjacent arcs in A' form a simplicial pair of D. So if two arcs of A' are adjacent, then they have the same tail and their heads are adjacent vertices. This implies two properties. First, if A'' is a subset of pairwise adjacent arcs of A', then the arcs in A'' have the same tail and span a clique. Second, the relation "a is adjacent with b" is an equivalence relation on A'. This relation defines a unique partition $\mathcal{S} = \{A'_1, \ldots, A'_k\}$ of A' such that the arcs in A'_i are pairwise adjacent, and that the A'_i's are pairwise disjoint. The first property implies that each A'_i is a simplicial star. So by definition, $\mathcal{S}$ is a simplicial stellar forest of D. □

2.3. Representing colorings of G by stable sets of $\tilde{G}$

The *line-graph of the digraph D* is the graph with vertex set $A(D)$ where two vertices are linked by an edge in the line-graph of D if they correspond to two adjacent arcs in D. In other words, we mean line-graph in the sense of undirected graphs. Let D be an acyclic orientation of the complementary graph $\overline{G}$ of G, so $V(D) = V(G)$ and $|A(D)| = \overline{m}$. The auxiliary graph $\tilde{G}$ is obtained from the line-graph of D by removing all edges between pairs of arcs which are simplicial in D. Equivalently, $\tilde{G}$ can be defined by choosing a total order $V(G) = \{v_1, \ldots, v_n\}$ (which induces an acyclic orientation D of $\overline{G}$) and letting $\tilde{G}$ be the auxiliary graph obtained from G as follows:

- $V(\tilde{G}) = \{a_1, \ldots, a_{\overline{m}}\} := \{\text{arcs } (v_i, v_j) \text{ such that } \{v_i, v_j\} \in E(\overline{G})$ and $i < j\}$.