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RESEARCH PAPER

Capital Asset Pricing Model with Generalized Elliptical Distribution

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Abstract: The distributions of returns on risky assets have an important effect on their equilibrium prices, and their empirical distributions have non-normal characteristics such as skewness and excess kurtosis, so assumption of normal distribution is not reasonable in capital asset pricing model (CAPM). Generalized elliptical distribution can well describe the empirically distributional characteristics of risky asset returns. This article assumes that risky asset returns have generalized elliptical distributions, proves Capital Asset Pricing Model with generalized elliptical distributions using the assumption of securities market economy and equilibrium analysis. As the generalized elliptical distribution can better describe the distributions of risky asset returns than normal distribution, CAPM with generalized elliptical distribution can better describe the behaviors of risky asset prices.

Key Words: capital asset pricing model; generalized elliptical distribution; normal distribution

1 Introduction

Sharpe, Lintner, and Mossin put forward the famous CAPM (Capital Asset Pricing Model)[1−3]. CAPM is based on strict assumptions about the economy and preferences of investors. Therefore, this model can not well describe the behaviors of asset prices in real world. After CAPM is put forward, economists study the problem of asset pricing by modifying the assumptions in CAPM, and asset pricing theory have improved very much. These advancements can be summarized to the following aspects:

(1) Keeping static analysis framework in CAPM, asset pricing problem was studied by modifying the assumption of frictionless market. For example, Black considered restricted riskless borrowing and lending and established a zero-beta CAPM; Mayers put forward a CAPM with non-marketable assets[4−5].

(2) Studying asset pricing problem by using the dynamic analysis framework in Merton and Lucas and the idea of consumption-based asset pricing in Breeden[6−8]. Among these researches, the most remarkable method was the establishment of new asset pricing models by modifying the utility functions of investors, such as asset pricing model based on preference for wealth[9], asset pricing model based on habit formation[10−12], asset pricing model based on preference for wealth and habit formation[13], asset pricing models using recursive utility function and stochastic differential utility[14−15], asset pricing model considering heterogeneous preference[16], asset pricing model considering lose aversion[17].

According to Central Limit Theorem, normal distribution is a good asymptotical distribution of risky asset return. And normal distribution has very good analytic properties. So, most asset pricing models use the following simplified assumption in common: assuming risky asset returns are normally distributed in static framework or assuming risky asset prices follow logarithmic Brown Motion in dynamic framework that means continuously compounded returns of risky asset are normally distributed.

However, many empirical researches have discovered that empirical distributions of risky asset returns have obvious no-normal characteristics, such as skewness and excess kurtosis. So the assumption that risky asset returns are normally distributed does not accord with the reality. Distributions of risky asset returns have a significant effect on decision-making of investors, consequently affect the equilibrium prices of risky assets when market is clearing, so the assumption of normal distribution in asset pricing models is reasonless and must be modified. Belkacem assumed that risky asset returns have stable distributions and proved CAPM with stable distribution[18]. However, the analytic properties of stable distribution are very bad, so Belkace had to use analysis method of portfolio frontier to prove his asset pricing model and there exist some mistakes in process of proof. Hamada and Valdez assumed that risky asset returns have elliptical distributions and proved CAPM with elliptical distributions by equilibrium analysis[19]. But elliptical distributions can only describe the characteristic of excess kurtosis, not the characteristic of skewness.

Frahm and Junker put forward generalized elliptical distribution which not only can describe the characteristics of skewness and excess kurtosis in the empirical distributions of risky asset returns, but also keeps the advantage of good analytic properties in normal distribution[20]. This article assumes that risky asset returns have generalized elliptical distributions and proves CAPM with generalized ellipti-

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cal distribution.

The structure of this article is as follow: the fist section is introduction; the second section introduces generalized elliptical distribution; the third section proves Stein formula with generalized elliptical distribution; the fourth section gives the assumptions of CAPM with generalized elliptical distribution; the fifth section proves CAPM with generalized elliptical distribution; the sixth section compares CAPM with generalized elliptical distribution to CAPM with normal distribution by a numerical example; the last section is conclusion.

2 Generalized elliptical distribution

Definition 2.1 *Let* $\boldsymbol{X} = (x_1, x_2, \cdots, x_n)'$ *be a* n*-dimension random vector. For some vector* $\boldsymbol{\mu} \in R^n$ *and some* $n \times n$ *semi-positive definite symmetric matrix* $\boldsymbol{\Sigma}$*, if characteristic function of* $\boldsymbol{X}$ *has following form:*

$$\boldsymbol{E}[\exp(i\boldsymbol{t'X})] = \exp(i\boldsymbol{t'\mu}) \cdot \phi(\boldsymbol{t'\Sigma t})$$

where $\phi(\cdot)$ *is a scalar function and called characteristic generator,* $\boldsymbol{t} \in R^n$*, then* $\boldsymbol{X}$ *has* n*-dimension elliptical distribution with parameters* $\boldsymbol{\mu}$*,* $\boldsymbol{\Sigma}$ *and* ϕ*, denoted as* $\boldsymbol{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$*.*

There is an important theorem about elliptical distribution.

Theorem 2.1 *A* n*-dimension random vector* $\boldsymbol{X}$ *has elliptical distribution* $E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ *(where rank of* $\boldsymbol{\Sigma}$ *is* k*) if and only if there exists a no-negative random variable* R *and* R *is independently distributed with random vector* $\boldsymbol{U}$ *which has a* k*-dimension uniform distribution on unit hyperplane* $\{\boldsymbol{z} \in R^k \mid \boldsymbol{z'z} = 1\}$*, and there exists* $n \times k$ *matrix* $\boldsymbol{A}$ *and* $\boldsymbol{AA'} = \boldsymbol{\Sigma}$*, such that* $\boldsymbol{X} =_d \boldsymbol{\mu} + R\boldsymbol{AU}$ *where* $=_d$ *denotes identical distribution.*

Elliptical distribution is a symmetric distribution family and many distributions are its special cases. For example, when characteristic generator $\phi(\cdot)$ is $\phi(x) = \exp(-x/2)$, $\boldsymbol{X}$ is normally distributed. Elliptical distribution has many good analytic properties and can describe excess kurtosis. Bradley and Taqqu and Embrechts, Lindskog and McNeil introduced properties of elliptical distribution in detail [21−22].

Elliptical distribution can describe excess kurtosis, but cannot describe skewness. On the basis of Theorem 2.1, Frahm and Junker[20] put forward generalized elliptical distribution which can describe both skewness and excess kurtosis.

Definition 2.2 *A* n*-dimension random vector* $\boldsymbol{X}$ *has* n*-dimension generalized elliptical distribution if and only if there exist a no-negative random variable* R*, a random vector* $\boldsymbol{U}$ *having* k*-dimension uniform distribution on unit hyperplane, a* n*-dimension vector* $\boldsymbol{\mu}$ *and* $n \times k$ *matrix* $\sqrt{\boldsymbol{\Sigma}}$*, such that*

$$\boldsymbol{X} =_d \boldsymbol{\mu} + R\sqrt{\boldsymbol{\Sigma}}\boldsymbol{U}$$

Frahm and Junker[20] proved the following theorem, and gives the density function of generalized elliptical distribution.

Theorem 2.2 *A* n*-dimension random vector* $\boldsymbol{X}$ *has* n*-dimension generalized elliptical distribution with parameters* $\boldsymbol{\mu}$ *and* Σ *(*$|\sqrt{\Sigma}| \neq 0$*). Furthermore, assume the joint distribution of* R *and* $\boldsymbol{U}$ *is absolute continuous. So density function of* $\boldsymbol{X}$ *is*

$$f(\boldsymbol{X}) = |\boldsymbol{\Sigma}|^{-1/2} g_n\left((\boldsymbol{X} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{X} - \boldsymbol{\mu}); \boldsymbol{u}\right) \qquad (1)$$

where

$$\boldsymbol{u} = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{X} - \boldsymbol{\mu}) \Big/ \sqrt{(\boldsymbol{X} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{X} - \boldsymbol{\mu})}$$

and $g_n(\cdot)$ *has the following form:*

$$g_n(t; \boldsymbol{u}) = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \cdot t^{-(n-1)/2} \cdot p_{R|\boldsymbol{U}=\boldsymbol{u}}(\sqrt{t})$$

where $p_{R|\boldsymbol{U}=\boldsymbol{u}}(\cdot)$ *is conditional density function of* R *when* $\boldsymbol{U} = \boldsymbol{u}$*.*

In this article, it is described that a n-dimension random vector $\boldsymbol{X}$ has n-dimension generalized elliptical distribution with density function $g_n(\cdot)$ and the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are denoted by $\boldsymbol{X} \sim GE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$.

3 Stein formula with generalized elliptical distribution

For proving CAPM with generalized elliptical distribution, Stein formula will be used. This section proves the following lemma first, then proves Stein formula with generalized elliptical distribution using this lemma.

Lemma 3.1 *Assume random variable* x *has one-dimension generalized elliptical distribution and is denoted as* $x \sim GE_1(\mu, \sigma^2, g)$*,* $h(\cdot)$ *is a differentiable function satisfying* $\mathrm{E}[|h'(\cdot)|] < \infty$*. So*

$$\sigma^2 \mathrm{E}[h'(x)] = \mathrm{E}[h(x^*)(x^* - \mu)]$$

where $x^* \sim GE_1(\mu, \sigma^2, -g')$

Proof. According to Theorem 2.2, density generator of x is a function of $(x-\mu)^2/\sigma^2$, so density of x can be denoted as $f(x) = \sigma^{-1} g\left(\frac{1}{2}(x-\mu)^2/\sigma^2\right)$. Then

$$\begin{aligned} E[h'(x)] &= \int_{-\infty}^{+\infty} h'(x)\sigma^{-1} g\left(\frac{1}{2}(x-\mu)^2 / \sigma^2\right) \mathrm{d}x \\ &= \sigma^{-1} g\left(\frac{1}{2}(x-\mu)^2/\sigma^2\right) h(x)\Big|_{-\infty}^{+\infty} \\ &- \int_{-\infty}^{+\infty} h(x)\sigma^{-2}\frac{x-\mu}{\sigma} g'\left(\frac{1}{2}(x-\mu)^2/\sigma^2\right) \mathrm{d}x \end{aligned}$$

According to the properties of $h(\cdot)$ and $g(\cdot)$, the first term equals zero. So

$$\sigma^2 \mathrm{E}[h'(x)] = \int_{-\infty}^{+\infty} h(x)(x-\mu)\left(-\frac{1}{\sigma} g'\left(\frac{1}{2}(x-\mu)^2/\sigma^2\right)\right) \mathrm{d}x$$

Define $x^* \sim GE_1(\mu, \sigma^2, -g')$, then

$$\sigma^2 \mathrm{E}[h'(x)] = \mathrm{E}[h(x^*)(x^* - \mu)].$$