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Least squares estimator for non-ergodic Ornstein–Uhlenbeck processes driven by Gaussian processes

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ABSTRACT

The statistical analysis for equations driven by fractional Gaussian process (fGp) is relatively recent. The development of stochastic calculus with respect to the fGp allowed to study such models. In the present paper we consider the drift parameter estimation problem for the non-ergodic Ornstein–Uhlenbeck process defined as $dX_t = \theta X_t dt + dG_t,\ t \geq 0$ with an unknown parameter $\theta > 0$, where G is a Gaussian process. We provide sufficient conditions, based on the properties of G, ensuring the strong consistency and the asymptotic distribution of our estimator $\widetilde{\theta}_t$ of θ based on the observation $\{X_s,\ s \in [0, t]\}$ as $t \to \infty$. Our approach offers an elementary, unifying proof of Belfadli (2011), and it allows to extend the result of Belfadli (2011) to the case when G is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. We also discuss the cases of subfractional Brownian motion and bifractional Brownian motion.

1. Introduction

While the statistical inference of Ito's type diffusions has a long history, the statistical analysis for equations driven by fractional Gaussian process is relatively recent. The development of stochastic calculus with respect to the fGp has allowed to study such models. We will recall several approaches to estimate the parameters in fractional models but we mention that the list below is not exhaustive:

- The MLE approach in Kleptsyna and Le Breton (2002), Tudor and Viens (2007): In general the techniques used to construct maximum likelihood estimators (MLE) for the drift parameter are based on Girsanov's transforms for fBm and depend on the properties of the deterministic fractional operators (determined by the Hurst parameter) related to the fBm. In this case, the MLE is not easily computable.
- A least squares approach has been proposed in Hu and Nualart (2010): The study of the asymptotic properties of the estimator is based on certain criteria formulated in terms of the Malliavin calculus. In the ergodic case, the statistical inference for several fractional Ornstein–Uhlenbeck (fOU) models has been recently developed in the papers (Azmoodeh & Morlanes, 2013; Azmoodeh & Viitasaari, 2015; Cénac & Es-Sebaiy, 2015; El Onsy, Es-Sebaiy, & Viens, 2014; Hu & Nualart, 2010; Hu & Song, 2013). The case of non-ergodic fOU process of the first kind and of the second kind can be found in Belfadli, Es-Sebaiy, and Ouknine (2011) and El Onsy, Es-Sebaiy, and Tudor (2014) respectively.

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- Method of moments: A new idea has been provided in Es-Sebaiy and Viens (2015), to develop the statistical inference for stochastic differential equations related to stationary Gaussian processes by proposing a suitable criteria. This approach is based on the Malliavin calculus, and it makes in principle the estimators easier to be simulated. Moreover, as an application, the models discussed in Azmoodeh and Morlanes (2013), Azmoodeh and Viitasaari (2015), El Onsy et al. (2014), Hu and Nualart (2010) have been studied in Es-Sebaiy and Viens (2015) by using this approach.

In this paper, we consider the non-ergodic Ornstein–Uhlenbeck process $X = \{X_t, t \geq 0\}$ given by the following linear stochastic differential equation

$$X_0 = 0; \qquad dX_t = \theta X_t dt + dG_t, \quad t \geq 0, \tag{1.1}$$

where G is a Gaussian process and $\theta > 0$ is an unknown parameter.

A problem here is to estimate the parameter θ when one observes the whole trajectory of X. In the case when the process X has Hölder continuous paths of order $\delta \in (\frac{1}{2}, 1]$ we can consider the following least squares estimator (LSE)

$$\widehat{\theta}_t = \frac{\int_0^t X_s dX_s}{\int_0^t X_s^2 ds}, \quad t \geq 0, \tag{1.2}$$

as estimator of θ, where the integral with respect to X is a Young integral (Young, 1936) (see Appendix). The estimator $\widehat{\theta}_t$ is obtained by the least squares technique, that is, $\widehat{\theta}_t$ (formally) minimizes

$$\theta \longmapsto \int_0^t \left|\dot{X}_s - \theta X_s\right|^2 ds.$$

Moreover, using the formula (A.1) we can rewrite $\widehat{\theta}_t$ as follows,

$$\widehat{\theta}_t = \frac{X_t^2}{2\int_0^t X_s^2 ds}, \quad t \geq 0. \tag{1.3}$$

Motivated by (1.3) we propose to use, in the general case, the right hand side of (1.3) as a statistic to estimate the drift coefficient θ of Eq. (1.1). More precisely, we define

$$\widetilde{\theta}_t = \frac{X_t^2}{2\int_0^t X_s^2 ds}, \quad t \geq 0. \tag{1.4}$$

This estimator $\widetilde{\theta}_t$ may exist even if X does not have Hölder continuous paths of order $\delta \in (\frac{1}{2}, 1]$.

We shall provide sufficient conditions, based on the properties of G, under which the estimator $\widetilde{\theta}_t$ is consistent (see Theorem 2.1), and the limit distribution of $\widetilde{\theta}_t$ is a standard Cauchy distribution (see Theorem 2.2).

Examples of the Gaussian process G.

Fractional Brownian motion:

Suppose that the process G given in (1.1) is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. By assuming that $H > \frac{1}{2}$, Belfadli et al. (2011) studied the LSE $\widehat{\theta}_t$ which coincides, in this case, with $\widetilde{\theta}_t$ by Remark 2.1. In this paper, we extend the result of Belfadli et al. (2011) to the case $H \in (0, 1)$. Moreover, we offer an elementary proof (see Section 3.1).

Sub-fractional Brownian motion:

Assume that the process G given in (1.1) is a subfractional Brownian motion with parameter $H \in (0, 1)$. For $H > \frac{1}{2}$, using an idea of Belfadli et al. (2011), Mendy (2013) studied the LSE $\widehat{\theta}_t$ which also coincides with $\widetilde{\theta}_t$. But the proof of Lemma 4.3 in Mendy (2013) relies on a possibly awed technique because the passage from line −7 to −6 on page 671 does not allow to obtain the convergence of $E\left[\left(e^{-\theta t}\int_0^t e^{\theta s} dS_s^H\right)^2\right]$ as $t \to \infty$. In the present paper, we give a solution of this problem and we extend the result to $H \in (0, 1)$ (see Section 3.2).

Bifractional Brownian motion:

To the best of our knowledge there is no study of the problem of estimating the drift of (1.1) in the case when G is a bifractional Brownian motion with parameters $(H, K) \in (0, 1)^2$. Section 3.3 is devoted to this question.

2. Asymptotic behavior of the estimator

Let $G = (G_t, t \geq 0)$ be a continuous centered Gaussian process defined on some probability space $(\Omega, \mathcal{F}, P)$ (here, and throughout the text, we assume that $\mathcal{F}$ is the sigma-field generated by G). The following assumptions are required.

($\mathcal{H}1$) The process G has Hölder continuous paths of order $\delta \in (0, 1]$.
($\mathcal{H}2$) For every $t \geq 0$, $E\left(G_t^2\right) \leq ct^{2\gamma}$ for some positive constants c and γ.