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Laplace transform ordering of time to failure in age replacement models

M. Kayid [a,*], S. Izadkhah [b], S. Alshami [a]

[a] Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

[b] Department of Statistics, Faculty of Mathematics and Computing, Higher Education Complex of Bam, Bam, Kerman, Iran

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ABSTRACT

The concept of mean time to failure in age replacement models plays an important role in reliability and life testing. In this paper, based on the comparison of Laplace transform of time to failure in an age replacement model, we introduce and study a new stochastic order. This new order is stronger than the Laplace transform and the mean time to failure in the age replacement model orders. Some motivations and descriptions of the proposed stochastic order are provided. In addition, several characterizations and preservation properties of the new order under some well-known reliability operations are discussed. As a consequence, a new class of life distributions is proposed, and some of its reliability properties are investigated. Finally, the problem of testing exponentiality against such class is discussed and some numerical results are presented.

1. Introduction

In reliability theory, failure of a unit during actual operation is costly or dangerous. If the unit is characterized by a failure rate that increases with age, it may be reasonable to replace it before it has aged too greatly. A commonly considered replacement policy is the policy based on age which is in force if a unit is always replaced at the time of failure or t hours after its installation, whichever comes first. Formally, let $X_{[t]}$ denote the lifetime of an item in a service, assuming that the item (with lifetime X and survival function $\bar{F}$) is replaced by a new one (whose lifetime is equal to X in distribution) at failure, or t hours after installation, whichever comes first. Then, according to Barlow and Proschan (1965) the survival function of $X_{[t]}$ is given by

$$S_t(x) = \sum_{n=0}^{\infty} \left[\bar{F}(t)\right]^n \bar{F}(x-nt) I_{[nt,(n+1)t)}(x), \quad x \geq 0, \tag{1.1}$$

which is actually the probability that the item does not fail in service before time x. The model (1.1) is well-known as the age replacement model. The mean of the random variable $X_{[t]}$, i.e., $m(t) = E(X_{[t]})$ is called mean time to failure (MTTF) in

* Correspondence to: Department of Mathematics and Computer Science, Faculty of Science, Suez University, Suez 41522, Egypt.
E-mail address: drkayid@ksu.edu.sa (M. Kayid).

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the age replacement model, which is given by Kayid, Ahmad, Izadkhah, and Abouammoh (2013)

$$m(t) = \frac{\int_0^t \bar{F}(x)dx}{F(t)}, \quad t > 0. \tag{1.2}$$

On the other hand, one of the main objectives of statistics and probability is to compare the magnitude of random variables via some quantities of their probability distributions. In reliability theory, the researcher can express various forms of properties about the lifetime distributions in terms of their survival functions, hazard rate functions, reversed hazard functions, mean residual functions, and other characteristics of probability distributions. These methods are much more informative than those based only on few numerical characteristics of distributions such as the mean and the median. Comparisons of random variables based on such different functions are usually establishing partial orders among them which is well-known as "stochastic orders". The need to provide a more detailed comparison of two random quantities has been the origin of the theory of stochastic orders that has grown significantly during last years (Belzunce, 2010; Shaked & Shanthikumar, 2007). Stochastic comparisons of the age replacement models will be useful to choose between two lifetime distributions in order to prolong the expected time to an in-service failure.

As a useful notion in applied mathematics and engineering, Laplace transform is very important in many areas of probability and statistics (Feller, 1971). For a non-negative random variable X with distribution function F, its Laplace transform is defined as

$$L_X(s) = \int_0^\infty e^{-sx} dF(x), \quad s \geq 0.$$

Define

$$L_X^*(s) = \int_0^\infty e^{-sx} \bar{F}(x)dx, \quad s \geq 0. \tag{1.3}$$

If X is absolutely continuous, then $L_X^*(s) = [1 - L_X(s)]/s$. Two well-known stochastic orders that have been introduced and studied in reliability theory are the Laplace transform order and the mean time to failure order in age replacement models, whose definitions are recalled here. Let X and Y be two lifetime random variables having distribution functions F and G, respectively, and denote by $\bar{F}$ (f) and $\bar{G}$ (g) their respective survival (density) functions. A random variable X is said to be smaller than Y in the Laplace transform order (denoted as $X \leq_{LT} Y$) if $L_X(s) \geq L_Y(s)$, for all $s > 0$, or equivalently if $L_X^*(s) \leq L_Y^*(s)$, for all $s \geq 0$, i.e. if

$$\int_0^\infty e^{-sx} \bar{F}(x)dx \leq \int_0^\infty e^{-sx} \bar{G}(x)dx, \quad \text{for all } s \geq 0.$$

Applications, properties and interpretations of the Laplace transform order can be found in Ahmed and Kayid (2004), Alzaid, Kim, and Proschan (1991), Belzunce, Gao, Hu, and Pellerey (2004), Belzunce, Ortega, and Ruiz (1999) and Shaked and Shanthikumar (2007). The mean time to failure order in the age replacement model is introduced and studied by Asha and Nair (2010). The lifetime random variable X is said to be smaller than Y in the mean time to failure order (denoted by $X \leq_{MTTF} Y$) if

$$\frac{\int_0^t \bar{F}(x)dx}{F(t)} \leq \frac{\int_0^t \bar{G}(x)dx}{G(t)}, \quad \text{for all } t > 0.$$

Recently, Kayid et al. (2013) studied further this new order and provided several characterizations, preservation properties and applications in the context of reliability and life testing. The purpose of this paper is to introduce and study a new stochastic order based on the comparison of Laplace transform of time to failure in the age replacement models. It will be shown that this stochastic order is stronger than the Laplace transform and the mean time to failure orders. A new class of lifetime distributions called decreasing Laplace transform of time to failure (DLTTF) in the age replacement models is also introduced and studied. The rest of the paper is organized as follows: the precise definitions of some stochastic orders as well as some classes of life distributions which will be used in the sequel, are given in Section 2. In Section 3, the Laplace transform order of time to failure in the age replacement model is introduced and studied. Some relationships between the new order and some other well-known orders are investigated. In that section, several characterizations and preservation properties of the new stochastic order under some reliability operations are also discussed. In Section 4, we propose the DLTTF class of lifetime distributions and provide some characterizations and preservation properties of it in the context of reliability. In Section 5, a test statistic for testing exponentiality against the proposed class is discussed and then the power and the critical values for the proposed statistic are calculated. Finally, in Section 6, we give a brief conclusion and some remarks on the future of this research.

For convenience, throughout this paper, the term increasing is used instead of monotone non-decreasing and the term decreasing is used instead of monotone non-increasing. We also assumed that the random variables under consideration are absolutely continuous, having 0 as the common left end point of their supports, and their expectations are assumed to be finite wherever they appear.