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Convergence rates of spectral distributions of large dimensional quaternion sample covariance matrices

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ABSTRACT

In this paper, we study the convergence rates of empirical spectral distributions of large dimensional quaternion sample covariance matrices. Assume that the entries of $\mathbf{X}_n$ $(p \times n)$ are independent quaternion random variables with means zero, variances 1 and uniformly bounded sixth moments. Denote $\mathbf{S}_n = \frac{1}{n}\mathbf{X}_n\mathbf{X}_n^*$. Using Bai's inequality, we prove that the expected empirical spectral distribution (ESD) converges to the limiting Marčenko–Pastur distribution with the ratio of dimension to sample size $y_p = p/n$ at a rate of $O\left(n^{-1/2}a_n^{-3/4}\right)$ when $a_n > n^{-2/5}$ or $O\left(n^{-1/5}\right)$ when $a_n \le n^{-2/5}$, where $a_n = (1-\sqrt{y_p})^2$. Moreover, the rates for both the convergence in probability and the almost sure convergence are also established. The weak convergence rate of the ESD is $O\left(n^{-2/5}a_n^{-1/2}\right)$ when $a_n > n^{-2/5}$ or $O\left(n^{-1/5}\right)$ when $a_n \le n^{-2/5}$. The strong convergence rate of the ESD is $O\left(n^{-2/5+\eta}a_n^{-1/2}\right)$ when $a_n > \kappa n^{-2/5}$ or $O\left(n^{-1/5}\right)$ when $a_n \le \kappa n^{-2/5}$ for any $\eta > 0$ where κ is a positive constant.

1. Introduction

Let $\mathbf{A}$ be a $p \times p$ Hermitian matrix and denote its eigenvalues by $s_j,\ j = 1, 2, \ldots, p$. The empirical spectral distribution (ESD) of $\mathbf{A}$ is defined by

$$F^{\mathbf{A}}(x) = \frac{1}{p}\sum_{j=1}^{p} I\left(s_j \le x\right),$$

where $I(D)$ is the indicator function of an event D. Huge data sets with large dimension and large sample size lead to failure of the applications of the classical limit theorems. In recent decades, the theory of random matrices (RMT) has been actively developed which enables us to find the solutions to this issue.

The sample covariance matrix is one of the most important random matrices in RMT, which can be traced back to Wishart (1928). And so far, most researchers focus on the real or complex case. As the wide applications of quaternions and quaternion matrices in quantum physics, robot technology and artificial satellite attitude control, etc., it is necessary to study the quaternion sample covariance matrix. Let us take a quaternion principal component analysis algorithm for face

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recognition as an instance. This algorithm uses quaternion matrices to represent color images, then they need the quaternion sample covariance matrix in order to construct a quaternion feature space.

First of all, we review a series of important results on the real or complex sample covariance matrix. Marčenko and Pastur (1967) proved that the ESD of large dimensional complex sample covariance matrices tends to the M–P law $F_y(x)$ with the density function

$$f_y(x) = \begin{cases} \frac{1}{2\pi xy\sigma^2}\sqrt{(b-x)(x-a)}, & a \le x \le b, \\ 0, & \text{otherwise}, \end{cases}$$

where $a = \sigma^2\left(1-\sqrt{y}\right)^2$, $b = \sigma^2\left(1+\sqrt{y}\right)^2$, σ^2 is the scale parameter, and the constant y is the limiting ratio of dimension p to sample size n. If $y > 1$, $F_y(x)$ has a point mass $1 - 1/y$ at the origin. After the limiting spectral distribution (LSD) of the sample covariance matrices is found, two important problems arise. The first is the bound on extreme eigenvalues, and the second is the convergence rate of the ESD with respect to sample size. Yin, Bai, and Krishnaiah (1988) proved that the largest eigenvalue of the large dimensional real sample covariance matrix tends to $\sigma^2\left(1+\sqrt{y}\right)^2$, a.s.. Bai and Yin (1993) established the conclusion that the smallest eigenvalue of the large dimensional real sample covariance matrix strongly converges to $\sigma^2\left(1-\sqrt{y}\right)^2$. For convergence rate, since Bai (1993a) established a Berry–Essen type inequality, much work has been done (see Bai, 1993b, Bai, Hu, & Zhou, 2012, Bai, Miao, & Tsay, 1997, Bai, Miao, & Yao, 2003, Gotze & Tikhomirov, 2004, 2010, among others). Here the readers are referred to three books (Anderson, Guionnet, & Zeitouni, 2010; Bai & Silverstein, 2010; Mehta, 2004) for more details.

Next, we present some parallel results for the quaternion case. In Li, Bai, and Hu (2013), it was proved that the ESD of a large dimensional quaternion sample covariance matrix tends to the M–P law. From Li and Bai (2013), we have known the limits of extreme eigenvalues of quaternion sample covariance matrices. Convergence rates of the ESD of the quaternion sample covariance matrix are considered in this paper.

In what follows, we introduce some notations about matrix representations of quaternions. The quaternion base can be represented by four 2×2 matrices as

$$\mathbf{e} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $i = \sqrt{-1}$ denotes the imaginary unit. Thus, a quaternion can be written by a 2×2 complex matrix as

$$x = a\cdot\mathbf{e} + b\cdot\mathbf{i} + c\cdot\mathbf{j} + d\cdot\mathbf{k} = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \triangleq \begin{pmatrix} \lambda & \omega \\ -\overline{\omega} & \overline{\lambda} \end{pmatrix}$$

where the coefficients a, b, c, d are real. The conjugate of x is defined as

$$\bar{x} = a\cdot\mathbf{e} - b\cdot\mathbf{i} - c\cdot\mathbf{j} - d\cdot\mathbf{k} = \begin{pmatrix} a-bi & -c-di \\ c-di & a+bi \end{pmatrix} = \begin{pmatrix} \overline{\lambda} & -\omega \\ \overline{\omega} & \lambda \end{pmatrix}$$

and its norm as

$$\|x\| = \sqrt{a^2+b^2+c^2+d^2} = \sqrt{|\lambda|^2+|\omega|^2}.$$

More details can be found in Adler (1995), Finkelstein, Jauch, Schiminovich, and Speiser (1962), Kuipers (1999), Mehta (2004), So, Thompson, and Zhang (1994), Yin, Bai, and Hu (2014) and Zhang (1995, 1997). It is worth mentioning that any $n \times n$ quaternion matrix $\mathbf{Y}$ can be represented as a $2n \times 2n$ complex matrix $\psi(\mathbf{Y})$. Consequently, we can deal with quaternion matrices as complex matrices.

Finally, we introduce two tools which play a key role in establishing the convergence rates of the ESD. The first is Bai's inequality:

Lemma 1.1 (*Bai Inequality in Bai (1993a)*). *Let F be a distribution function and G be a function of bounded variation satisfying $\int |F(x) - G(x)|\mathrm{d}x < \infty$. Denote their Stieltjes transforms by $f(z)$ and $g(z)$, respectively, where $z = u + iv \in \mathbb{C}^+$. Then we have*

$$\begin{aligned}\|F-G\|_{KS} &\overset{\text{def}}{=} \sup_x |F(x)-G(x)| \\ &\le \frac{1}{\pi(1-\kappa)(2\gamma-1)}\left[\int_{-A}^{A}|f(z)-g(z)|\,\mathrm{d}u + 2\pi v^{-1}\int_{|x|>B}|F(x)-G(x)|\,\mathrm{d}x \right.\\ &\quad \left. + v^{-1}\sup_x\int_{|s|\le 2va}|G(x+s)-G(x)|\,\mathrm{d}s\right], \qquad (1.1)\end{aligned}$$

where a, γ, A and B are positive constants such that $A > B$,

$$\gamma = \frac{1}{\pi}\int_{|u|<a}\frac{1}{u^2+1}\mathrm{d}u > \frac{1}{2}, \quad \textit{and} \quad \kappa = \frac{4B}{\pi(A-B)(2\gamma-1)} < 1.$$