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Comparisons of smallest order statistics from Weibull distributions with different scale and shape parameters

Nuria Torrado

Centre for Mathematics, University of Coimbra, Apartado 3008, EC Santa Cruz, 3001-501 Coimbra, Portugal

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A B S T R A C T

Weibull distribution is a very flexible family of distributions which has been applied in a vast number of disciplines. In this work, we investigate stochastic properties of the smallest order statistics from two independent heterogeneous Weibull random variables with different scale and shape parameters. Furthermore, we study the hazard rate order of the smallest order statistics from lower-truncated Weibull distributions due to, in general, Weibull random variables are not ordered according to this ordering in the shape parameter.

1. Introduction

Weibull distribution is a very flexible family of distributions commonly used in modeling lifetime data. This distribution is a member of the family of extreme value distributions, of the scale models and also, of the proportional hazard models. Weibull distribution has been applied in nearly all scientific disciplines, such as engineering, physics, chemistry, meteorology, medicine, economics, inventory control, biology, etc. (see chapter 7 in Rinne, 2009, for more details).

Let $W(\alpha, \lambda)$ denote Weibull random variables with shape parameter α and scale parameter λ, both positive. Suppose $X_1, \ldots, X_n$ are independent Weibull random variables with $X_i \sim W(\alpha_i, \lambda_i)$, $i = 1, \ldots, n$. We denote by $X_{1:n}$ the smallest order statistic from $X_1, X_2, \ldots, X_n$. Then, its survival function is given by

$$\bar{F}_{1:n}(t) = \prod_{i=1}^{n} \bar{F}_i(t),$$

its density function is

$$f_{1:n}(t) = \prod_{i=1}^{n} \bar{F}_i(t) \sum_{i=1}^{n} h_i(t), \tag{1.1}$$

and its hazard rate function is

$$h_{1:n}(t) = \sum_{i=1}^{n} h_i(t). \tag{1.2}$$

E-mail address: nuria.torrado@gmail.com.

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Khaledi and Kochar (2006) proved that smallest order statistics from Weibull distributions with a common shape parameter and with scale parameters as $(\lambda_1, \ldots, \lambda_n)$ and $(\theta_1, \ldots, \theta_n)$ are ordered in the hazard rate order if one vector of scale parameters majorizes the other one. Similar result for the reversed hazard rate order is also obtained by Fang and Zhang (2013). Here, we strengthening these results to the likelihood ratio order. Further results for the largest order statistics from Weibull distributions can be found in Fang and Zhang (2012); Khaledi, Farsinezhad, and Kochar (2011) and Torrado and Kochar (2015).

In this article, we will discuss stochastic comparisons of the smallest order statistics arising from Weibull distributions with different both shape and scale parameters. First, we provide an example to illustrate potential application of this research. In the context of reliability theory, consider a series system with n components. Let X_i, $i = 1, \ldots, n$, denote the failure time of component i, and assume that X_i's are independent Weibull random variables with X_i having its shape parameter α_i and scale parameter λ_i. As a result, $X_{1:n} = \min\{X_1, \ldots, X_n\}$, the minimum from independent heterogeneous Weibull random variables, is the lifetime of such a series system. It would be of great interest to investigate the stochastic properties of $X_{1:n}$.

In general, the Weibull distribution is not hazard rate ordered in the shape parameter α, however, we give conditions under which the smallest order statistics from multiple-outlier Weibull models are ordered according to the hazard rate order. Let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier Weibull model with shape parameter α_1 for $i = 1, \ldots, p$ and α_2 for $i = p + 1, \ldots, n$, and scale parameters

$$\left(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}\right),$$

where $p + q = n$, and $Y_1, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier Weibull model with shape parameter α_1 for $i = 1, \ldots, p$ and α_2 for $i = p + 1, \ldots, n$, and scale parameters

$$\left(\underbrace{\theta_1, \ldots, \theta_1}_{p}, \underbrace{\theta_2, \ldots, \theta_2}_{q}\right).$$

Suppose $\min(\lambda_1, \lambda_2) \le \min(\theta_1, \theta_2) \le \max(\lambda_1, \lambda_2) \le \max(\theta_1, \theta_2)$. It is established here that, for any $\alpha_1, \alpha_2 > 0$, smallest order statistics from two multiple-outlier Weibull models with different scale and shape parameters are ordered according to hazard rate ordering.

We also study stochastic comparisons among the smallest order statistics from lower-truncated Weibull random variables. Truncated Weibull distributions have been used in reliability theory, see Zhang and Xie (2011) and the references cited therein. Let $LTW(\alpha, \lambda)$ denote lower-truncated Weibull random variables with shape parameter α, scale parameter λ and support $[1/\lambda, \infty)$. Let X_1 and X_2 be two independent random variables such that $X_i \sim LTW(\alpha_i, \lambda_i)$, $i = 1, 2$, and let Y_1^* and Y_2^* another set of independent random variables with $Y_i^* \sim LTW(\alpha_i^*, \theta_i)$, $i = 1, 2$. Denote by $X_{1:2}$ and $Y_{1:2}^*$ the corresponding smallest order statistic from X_i's and Y_i^*'s, respectively. It is proved that, among others, if (θ_1, θ_2) majorize (λ_1, λ_2) and $\left(\alpha_1^*, \alpha_2^*\right)$ majorize (α_1, α_2), then $X_{1:2}$ is stochastically larger than $Y_{1:2}^*$ in the sense of the hazard rate ordering. In addition, we extend this result to multiple-outlier lower-truncated Weibull models.

The rest of the article is organized as follows. Section 2 is devoted to review some definitions of stochastic orderings which will be used in the sequel. In Section 3, we present our main results about stochastic comparisons among smallest order statistics from Weibull distributions with different both shape and scale parameters. Because, in general, Weibull random variables is not hazard rate ordered in the shape parameter, we investigate in Section 4, the hazard rate ordering between the minimum of lower-truncated Weibull random variables with different both shape and scale parameters.

2. Definitions

In this section, we first recall some notions of stochastic and majorization orderings. Throughout the article the terms *increasing* and *decreasing* stand for *non-decreasing* and *non-increasing*, respectively.

Let X and Y be univariate random variables with cumulative distribution functions (c.d.f.'s) F and G, survival functions $\bar{F}\ (=1 - F)$ and $\bar{G}\ (=1 - G)$, p.d.f.'s f and g, hazard rate functions $h_F\ \left(=f/\bar{F}\right)$ and $h_G\ \left(=g/\bar{G}\right)$, and reverse hazard rate functions $r_F\ (=f/F)$ and $r_G\ (=g/G)$, respectively. The following definitions introduce stochastic orders, which are considered in this article, to compare the magnitudes of two random variables. For more details on stochastic comparisons, see Shaked and Shanthikumar (2007).

Definition 2.1. We say that X is smaller than Y in the:

(a) usual stochastic order, denoted by $X \le_{\rm st} Y$, if $\bar{F}(x) \le \bar{G}(x)$ for all x,
(b) hazard rate order, denoted by $X \le_{\rm hr} Y$, if $h_F(x) \ge h_G(x)$ for all x,
(c) reversed hazard rate order, denoted by $X \le_{\rm rh} Y$, if $r_F(x) \le r_G(x)$ for all x,
(d) likelihood ratio order, denoted by $X \le_{\rm lr} Y$, if $g(x)/f(x)$ is increasing in x for all x for which the ratio is well defined.