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Asymptotic properties for distributions and densities of extremes from generalized gamma distribution ☆

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ABSTRACT

For a generalized gamma random sequence, some fundamental properties and convergence rates of the distribution of its partial maximum to the Gumbel extreme value distribution are derived. The asymptotic expansions for the distribution and density of maximum from generalized gamma distribution are given under an optimal choice of normalizing constants.

1. Introduction

The three parameter generalized gamma distribution (denoted by GGD) studied by Stacy (1962) is the most general form of the gamma distribution. A random variable X is said to have a GGD with scale parameter $\lambda > 0$ and shape parameters $\beta > 0, c > 0$ if its probability density function (pdf) is given by

$$f(x) = \frac{c\lambda^{c\beta}}{\Gamma(\beta)} x^{c\beta-1} \exp\{-(\lambda x)^c\}, \quad x > 0, \tag{1.1}$$

where $\Gamma(\cdot)$ denotes the gamma function (Nelson, 1991). Noting that $\beta = 1, c = 1$ and $\beta = \frac{1}{2}, c = 2$ respectively stand for the Weibull, gamma and half-normal distributions. In addition, the log-normal distribution is a limiting special case when $\beta \to \infty$ (cf. Guass, Fredy, Lourdes, & Mário, 2013).

The gamma distribution has been widely used in kinds of areas, such as engineering, hydrology and survival analysis. In particular, gamma distribution plays a prominent role in actuarial science since most total insurance claim distributions have roughly the same shape as gamma distributions: skewed to the right, non-negatively supported and unimodal (Furman, 2008). In addition, gamma distributions have been studied in applying gamma approximations for modeling insurance portfolios, see Hürlimann (2001) and Rioux and Klugman (2006). Also, by using the translated gamma distribution, Kaas, Goovaerts, Dhaene, and Denuit (2001) modeled the total claims on a number of policies in the individual risk model and Bowers, Gerber, Hickman, Jones, and Nesbitt (1997) established a model for the aggregate insurance claims.

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Throughout the paper, let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with the cumulative distribution function (cdf) F following the GGD ($F \sim GGD(\beta, \lambda, c)$ for short, where $\lambda > 0$, $\beta > 0$, and $c > 0$), and let $M_n = \bigvee_{i=1}^{n} X_i$ denote the maximum. If there exist normalizing constants $a_n > 0$, $b_n \in R$ and nondegenerate cdf G such that, for each continuity point x of G,

$$\lim_{n\to\infty} P(M_n \leq a_n x + b_n) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x), \tag{1.2}$$

then $G(x)$ must be one of the three classical extreme value distributions. It is well known that G can be only one of three types of extreme value distribution functions, namely:

$$\Phi_\alpha(x) = \begin{cases} 0, & \text{if } x < 0, \\ \exp\{-x^{-\alpha}\}, & \text{if } x \geq 0, \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases}$$

$$\Lambda(x) = \exp\{-e^{-x}\}, \quad x \in \mathbb{R},$$

where α is a positive parameter. More details see Resnick (1987).

The aim of this note is to study some fundamental properties like the Mills type inequality, Mills ratio and the distributional tail representation of GGD(β, λ, c), and establish the convergence rate, asymptotic expansions for distribution and density of the maxima for samples obeying GGD(β, λ, c).

For the fundamental properties, which decide the max-domain of attractions the distribution belongs to, we refer to Liao, Peng, Nadarajah, and Wang (2014), Peng and Lin (2010) and Peng, Tong, and Nadarajah (2009) for the general error distribution (GED), short tailed symmetric distribution and the skew normal, respectively.

In extreme value theory, the quality of convergence of normalized maximum of a sample is an interesting topic in recent literature. For the convergence rate of (1.2), we refer to Leadbetter, Lindgren, and Rootzén (1983) and de Haan and Resnick (1996) for general cases and for specific cases, we refer to Hall (1979) for normal distribution, Peng, Nadarajah, and Lin (2010) for GED distribution and Liao and Peng (2012) for log-normal samples. Nair (1981) derived the asymptotic expansions of extremes of normal samples and Liao et al. (2014) extended the result to skew normal samples. For the related material convergence in various models of the density, the density of Λ case and Φ_α case has been considered by de Haan and Resnick (1982) and Pickands (1967), respectively. de Haan and Resnick also proved a local limit theorem for sample extremes and showed that the density of the normalized maximum converges to the limit density under Von Mises conditions in the L_p-metric, for more culminate efforts about the density convergence, see Omey (1988) and Sweeting (1985).

The contents of this note are arranged as follows: Section 2 derives some preliminary properties related to GGD. Section 3 obtains the pointwise convergence rate of the maximum of GGD. The asymptotic expansions for distributions and densities of maximum from the GGD sample and the proofs are given in Section 4.

2. Preliminary results

In this section, we derive some preliminary but important properties about GGD(β, λ, c). These properties imply that GGD(β, λ, c) belongs to the max-domain of attraction of the Gumbel extreme value distribution.

Firstly, we derive the Mills equalities and Mills ratio of GGD(β, λ, c), which are stated as follows.

Proposition 2.1. *Let $F(x)$ and $f(x)$ denote the cdf and pdf of GGD(β, λ, c), respectively. For $\lambda > 0$ and $c > 0$, we have*

(i) *for $0 < \beta \leq 1$ and $x > \lambda^{-1}[(\beta-1)(\beta-2)]^{1/2c}$,*

$$\frac{x^{1-c}}{c\lambda^c}\left(1 + \frac{\beta-1}{(\lambda x)^c}\right) \leq \frac{1-F(x)}{f(x)} \leq \frac{x^{1-c}}{c\lambda^c}\left(1 - \frac{(\beta-1)(\beta-2)}{(\lambda x)^{2c}}\right)^{-1}. \tag{2.1}$$

(ii) *for $1 < \beta < 2$ and $x > 0$,*

$$\frac{x^{1-c}}{c\lambda^c}\left(1 + \frac{\beta-1}{(\lambda x)^c}\right)\left(1 - \frac{(\beta-1)(\beta-2)}{(\lambda x)^{2c}}\right)^{-1} \leq \frac{1-F(x)}{f(x)} < \frac{x^{1-c}}{c\lambda^c}\left(1 + \frac{\beta-1}{(\lambda x)^c}\right). \tag{2.2}$$

(iii) *for $\beta \geq 2$ and $x > \lambda^{-1}[(\beta-1)(\beta-2)]^{1/2c}$,*

$$\frac{x^{1-c}}{c\lambda^c}\left(1 + \frac{\beta-1}{(\lambda x)^c}\right) \leq \frac{1-F(x)}{f(x)} \leq \frac{x^{1-c}}{c\lambda^c}\left(1 + \frac{\beta-1}{(\lambda x)^c}\right)\left(1 - \frac{(\beta-1)(\beta-2)}{(\lambda x)^{2c}}\right)^{-1}. \tag{2.3}$$