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Non-central limit theorem of the weighted power variations of Gaussian processes[☆]

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ABSTRACT

By using the techniques of the Malliavin calculus, we investigate the asymptotic behavior of the weighted q-variations of continuous Gaussian process of the form $B_t = \int_0^t K(t,s)\, dW(s)$, where W is the standard Brownian motion and K is a square integrable kernel. In particular, in the case of fractional Brownian motion with the Hurst parameter H, the limit can be expressed as the sum of $q+1$ Skorohod integrals of the Hermite process with self-similarity $q(H-1)+1$. This result gives the relation between the Skorohod integral and a pathwise Young integral of the Hermite process.

1. Introduction

When we investigate the behavior on a path of stochastic processes; their power variations are often studied. There exists a very extensive study on the subject. Recently in several works, the asymptotic behavior on the weighted power variations of a fractional Brownian motion has been studied by using the Malliavin calculus (see Nourdin, 2008, Nourdin & Nualart, 2010 and Nourdin, Nualart, & Tudor, 2010). Chronopoulou, Tudor, and Viens (2008) and Tudor and Viens (2009) uncovered the asymptotic behavior on the quadratic variations of non-Gaussian processes known as the Rosenblatt process and other Hermite process, which are a generalization of fractional Brownian motion.

In particular, Nourdin et al. (2010) proved the following non-central limit theorem of renormalized weighted power variations of order $q \geq 2$ of the fractional Brownian motion B^H with the Hurst parameter $H > 1 - \frac{1}{2q}$:

$$n^{q-1} \frac{1}{q!} \sum_{j=0}^{n-1} f\left(B^H_{\frac{j}{n}}\right) I_q\left(\mathbf{1}^{\otimes q}_{\left[\frac{j}{n}, \frac{j+1}{n}\right]}\right) \xrightarrow{L^2} \int_0^1 f(B^H_s) \circ dZ^{(q)}_{H'}(s), \tag{1}$$

where I_q is the multiple stochastic integral with respect to the fractional Brownian motion B^H and $Z^{(q)}_{H'}$ denotes the Hermite process of order q with self-similarity $H' = q(H-1)+1$. Here we observe that the stochastic integral $\int_0^1 f(B^H_s) \circ dZ^{(q)}_{H'}(s)$

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is a pathwise Young integral. For an integer $q \geq 2$, we write H_q for the Hermite polynomial with order q. The relation (5) between multiple stochastic integrals and Hermite polynomials implies that the left-hand side in (1) can be expressed as (see Section 2)

$$n^{q-1}\frac{1}{q!}\sum_{j=0}^{n-1} f\left(B^H_{\frac{j}{n}}\right) I_q\left(\mathbf{1}^{\otimes q}_{\left[\frac{j}{n},\frac{j+1}{n}\right]}\right) = n^{q-1}\sum_{j=0}^{n-1} f\left(B^H_{\frac{j}{n}}\right) H_q\left(B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}\right). \tag{2}$$

Suppose that the process $B = \{B_t, t \in [0, 1]\}$ is a centered continuous Gaussian process of the form

$$B_t = \int_0^t K(t, x)dW(x), \tag{3}$$

where W is the standard Brownian motion and K is a square integrable kernel. Then the covariance function $R(s, t)$ of the continuous Gaussian process B can be expressed as

$$R(t, s) = \int_0^{s\wedge t} K(t, x)K(s, x)dx, \tag{4}$$

where the kernel satisfying $\sup_{t\in[0,1]} \int_0^t K(t, x)^2 dx < \infty$.

Breuer and Major (1983) proved that the central limit theorem holds for functional with the Hermite rank $q \geq 1$ of stationary Gaussian fields if the correlation function r satisfies the condition $\sum_n |r(n)|^q < \infty$. This is called the celebrated Breuer–Major theorem. In this paper, we prove the non-central limit theorem of the weighted power (or Hermite) variation of the Gaussian process B of the form (3) including the fractional Brownian motion and show the L^2 convergence to the limit which can be expressed as the sum of $q + 1$ iterated Skorohod integrals with respect to a standard Brownian motion. In particular, when $B = B^H$, we obtain, from the result of (1) in Nourdin et al. (2010), the relation between Skorohod integral and the pathwise Young integral of the Hermite process of order $q - k$, $0 \leq k \leq q$.

2. Preliminaries

The main tool in this paper is the Malliavin calculus. In this section, we recall some basic facts about the Malliavin calculus for Gaussian processes. For a more detailed reference, see Nualart (2006). Suppose that $\mathcal{H}$ is a real separable Hilbert space with scalar product denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Let $B = \{B(h), h \in \mathcal{H}\}$ be an isonormal Gaussian process that is a centered Gaussian family of random variables such that $\mathbb{E}[B(h)B(g)] = \langle h, g\rangle_{\mathcal{H}}$. For every $n \geq 1$, let $\mathbb{H}_n$ be the nth Wiener chaos of B that is the closed linear subspace of $\mathbb{L}^2(\Omega)$ generated by $\{H_n(B(h)) : h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where H_n is the nth Hermite polynomial. We define a linear isometric mapping $I_n : \mathcal{H}^{\odot n} \to \mathbb{H}_n$ by

$$I_n(h^{\otimes n}) = n!H_n(B(h)), \tag{5}$$

where $\mathcal{H}^{\odot n}$ is the symmetric tensor product.

Let $\mathcal{S}$ be the class of smooth and cylindrical random variables F of the form

$$F = f(B(\varphi_1), \ldots, B(\varphi_n)), \tag{6}$$

where $n \geq 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ and $\varphi_i \in \mathcal{H}$, $i = 1, \ldots, n$. Let us denote by D the Malliavin derivative for B and δ by the adjoint operator of D. By iteration, we can define the lth Malliavin derivative D^l for B. Let us denote by δ^l the adjoint operator of the lth Malliavin derivative D^l. Then we call δ^l the multiple Skorohod integral of order l with respect to B.

We denote by $\mathbb{D}^{l,p}$ the closure of its associated smooth random variable class with respect to the norm

$$\|F\|^p_{l,p} = \mathbb{E}(|F|^p) + \sum_{k=1}^{l} \mathbb{E}(\|D^k F\|^p_{\mathcal{H}^{\otimes k}}).$$

The domain of δ^l, denoted by $\mathrm{Dom}(\delta^l)$, is an element $u \in \mathbb{L}^2(\Omega; \mathcal{H}^{\otimes l})$ such that

$$|\mathbb{E}(\langle D^l F, u\rangle_{\mathcal{H}^{\otimes l}})| \leq C(\mathbb{E}|F|^2)^{1/2} \quad \text{for all } F \in \mathbb{D}^{l,2}.$$

The following property given in Lemma 2.1 of Nourdin and Nualart (2010) will be used.

Lemma 1. *Let $q \geq 1$ be an integer. Suppose that $F \in \mathbb{D}^{q,2}$ and u is a symmetric in $\mathrm{Dom}(\delta^q)$. Assume that for any $0 \leq r + l \leq q$, $\langle D^r F, \delta^l(u)\rangle_{\mathcal{H}^{\otimes r}} \in \mathbb{L}^2(\Omega; \mathbb{H}^{\otimes q-r-l})$. Then for any $r = 0, \ldots, q - 1$, $\langle D^r F, u\rangle_{\mathcal{H}^{\otimes r}} \in \mathrm{Dom}(\delta^{q-r})$ and*

$$F\delta^q(u) = \sum_{k=0}^{q} \binom{q}{k} \delta^{q-k}\Big(\langle D^k F, u\rangle_{\mathcal{H}^{\otimes k}}\Big). \tag{7}$$

3. Main result

Let us set

$$\varrho_n(j) = \int_{\frac{j}{n}}^{\frac{j+1}{n}} K\left(\frac{j+1}{n}, x\right)^2 dx \quad \text{and} \quad \vartheta_n(j) = \int_0^{\frac{j}{n}} \left|K\left(\frac{j+1}{n}, x\right) - K\left(\frac{j}{n}, x\right)\right|^2 dx.$$