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Asymptotic behavior of the weighted cross-variation with respect to fractional Brownian sheet☆

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ABSTRACT

By using the techniques of Malliavin calculus, we investigate the asymptotic behavior of the weighted cross-variation of fractional Brownian sheet with the case when Hurst parameter $H = (H_1, H_2)$ belongs to $(0, 1/2) \times (1/2, 1)$ or $(1/2, 1) \times (0, 1/2)$.

1. Introduction

Recently in several works, the asymptotic behavior on the weighted power variations of a fractional Brownian motion (fBm) has been studied by using Malliavin calculus (see Nourdin, 2008, Nourdin & Nualart, 2010 and Nourdin, Nualart, & Tudor, 2010). For the two-parameter processes, a central limit theorem has been obtained in Réveillac (2009a) for the weighted quadratic variations of a standard Brownian sheet. Furthermore, Réveillac in Réveillac (2009b) proved a central limit theorem for the finite-dimensional laws of the weighted quadratic variations of fractional Brownian sheet (fBs).

In this paper, we study the asymptotic behavior on the weighted cross-variation for fBs $B^H = (B^H_{s,t}, (s, t) \in [0, 1]^2)$ with Hurst parameters $H = (H_1, H_2)$, $0 < H_1, H_2 < 1$. More precisely, we will consider the sequence $\{F_{n,m}, n, m \geq 1\}$ given by

$$F_{n,m} = \sum_{i,k=1}^{2^n} \sum_{j,l=1}^{2^m} \mathbf{1}_{[i \leq k-1, l \leq j-1]} f\left(B^H_{\frac{k-1}{2^n}, \frac{j-1}{2^m}}\right) \Delta^{n,m}_{i,j}(B^H) \Delta^{n,m}_{k,l}(B^H), \tag{1}$$

where $f : R \to R$ is a smooth function and

$$\Delta^{n,m}_{i,j}(B^H) = B^H_{\frac{i}{2^n}, \frac{j}{2^m}} - B^H_{\frac{i}{2^n}, \frac{j-1}{2^m}} - B^H_{\frac{i-1}{2^n}, \frac{j}{2^m}} + B^H_{\frac{i-1}{2^n}, \frac{j-1}{2^m}}.$$

The sequence $\{F_{n,m}\}$ of the type given by (1) is to be necessary for the theory of stochastic calculus for two-parameter processes such as standard Brownian sheet or fBs (see Cairoli & Walsh, 1975 and Wong & Zakai, 1974 for standard Brownian

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sheet, and see Kim, Jeon, & Park, 2008 for fBs). The sequence $F_{n,m}$ can be written as

$$F_{n,m} = \sum_{k=1}^{2^n}\sum_{l=1}^{2^m} f\left(B^H_{\frac{k-1}{2^n},\frac{l-1}{2^m}}\right)\left(B^H_{\frac{k}{2^n},\frac{l-1}{2^m}} - B^H_{\frac{k-1}{2^n},\frac{l-1}{2^m}}\right)\left(B^H_{\frac{k-1}{2^n},\frac{l}{2^m}} - B^H_{\frac{k-1}{2^n},\frac{l-1}{2^m}}\right). \quad (2)$$

In the proof of Itô's formula for fBs with Hurst parameters $H_1, H_2 > \frac{1}{2}$ in Tudor and Viens (2003) (or see Kim et al., 2008), Tudor and Viens (2003) show that as n and m tend to infinity, the sequence $\{F_{n,m}\}$ of the form (2) converges, in $L^2(\Omega)$, to

$$H_1H_2\int_0^1\int_0^1 f(B^H_{s,t})s^{2H_1-1}t^{2H_2-1}dsdt + \int_0^1\int_0^1 f(B^H_{s,t})d\tilde{M}_{s,t} + H_1\int_0^1\int_0^1 f'(B^H_{s,t})s^{2H_1-1}t^{2H_2}dsdtB^H_{s,t}$$
$$+H_2\int_0^1\int_0^1 f'(B^H_{s,t})s^{2H_1}t^{2H_2-1}dsB^H_{s,t}dt + H_1H_2\int_0^1\int_0^1 f(B^H_{s,t})s^{4H_1-1}t^{4H_2-1}dsdt. \quad (3)$$

See Tudor and Viens (2003) for the precise definition of the various stochastic integrals appearing in (3).

Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ be two points in the rectangle $R_1 = [0, 1]^2$. The notation $\mathbf{a} \otimes \mathbf{b}$ denotes the point (a_1, b_2), and $\mathbf{a} \trianglelefteq_1 \mathbf{b}$ denotes the condition $a_1 \le b_1$ and $a_2 \ge b_2$. By using the definition of stochastic integrals of various types in Kim et al. (2008), we can show that the sequence $\{F_{n,m}\}$ of the form (1) converges, in $L^2(\Omega)$, to

$$\int_{R_1} f(B^H_{\mathbf{a}})d\mu_H(\mathbf{a}) + \int_{R_1\trianglelefteq_1 R_1} f(B^H_{\mathbf{b}\otimes\mathbf{a}})dB^H_{\mathbf{a}}dB^H_{\mathbf{b}} + \int_{R_1\trianglelefteq_1 R_1} f'(B^H_{\mathbf{b}\otimes\mathbf{a}})d\mu_H(\mathbf{a})dB^H_{\mathbf{b}} + \int_{R_1\trianglelefteq_1 R_1} f'(B^H_{\mathbf{b}\otimes\mathbf{a}})dB^H_{\mathbf{a}}d\mu_H(\mathbf{b})$$
$$+\int_{R_1\trianglelefteq_1 R_1} f''(B^H_{\mathbf{b}\otimes\mathbf{a}})d(\mu_H\otimes\mu_H)(\mathbf{a},\mathbf{b}), \quad (4)$$

where the measure μ_H is given by $d\mu_H(\mathbf{a}) = 2H_1H_2a_1^{2H_1-1}a_2^{2H_2-1}da_1da_2$, and the notation $R_1 \trianglelefteq_1 R_1$ denotes the region $\{(\mathbf{a}, \mathbf{b}) \in R_1 \times R_1 : \mathbf{a} \trianglelefteq_1 \mathbf{b}\}$.

In the case of fBm $B^H = (B^H, t \in [0, 1])$ with Hurst parameter $H \in (0, 1)$, if $H < \frac{1}{2}$, the quadratic variation of fBm is infinite, but if $H > \frac{1}{2}$, the quadratic variation of fBm is zero. In this context, we have an interest in the asymptotic behaviors for the normalized sequence of the form (1) in the case when $(H_1, H_2) \in (0, 1/2) \times (1/2, 1)$ and $(H_1, H_2) \in (1/2, 1) \times (0, 1/2)$.

The purpose of the present work is to prove the following result:

A: The function f belongs to $\mathcal{C}^4(R)$ and $\sup_{s,t\in[0,1]} \mathbb{E}[|f^{(i)}(B^H_{s,t})|^p] < \infty$ for any $p \in (0, \infty)$ and $i = 0, \ldots, 4$.

Theorem 1. *Let $B^H = (B^H_{s,t}, (s, t) \in [0, 1]^2)$ be fBs with Hurst parameter $H = (H_1, H_2)$. Then we have*

(i) *If $f : R \to R$ satisfies the assumption* **A** *and if $(H_1, H_2) \in (0, 1/2) \times (1/2, 1)$, then we have, as $n \to \infty$,*

$$F^{(1)}_{n,m} = \frac{2^{2nH_1}}{2^n}\sum_{i,k=1}^{2^n}\sum_{j,l=1}^{2^m}\mathbf{1}_{[i\le k-1,l\le j-1]}f\left(B^H_{\frac{k-1}{2^n},\frac{j-1}{2^m}}\right)\left[\Delta^{n,m}_{i,j}(B^H)\Delta^{n,m}_{k,l}(B^H) - \frac{1}{4(k-1)(j-1)2^{2nH_1+2mH_2}}\right]$$
$$\xrightarrow{L^2} \int_{R_1\trianglelefteq_1 R_1} f''(B^H_{\mathbf{b}\otimes\mathbf{a}}) \circ dB^H_{\mathbf{a}}d\mu_{(1/2,H_2)}(\mathbf{b}), \quad (5)$$

where $d\mu_{(1/2,H_2)}(\mathbf{b}) = H_2b_2^{2H_2-1}db_1db_2$ and the integration with respect to $B^H_{\mathbf{a}}$ is a path-wise Young integral.

(ii) *If $f : R \to R$ satisfies the assumption* **A** *and if $(H_1, H_2) \in (1/2, 1) \times (0, 1/2)$, then we have, as $n \to \infty$,*

$$F^{(2)}_{n,m} = \frac{2^{2mH_2}}{2^m}\sum_{i,k=1}^{2^n}\sum_{j,l=1}^{2^m}\mathbf{1}_{[i\le k-1,l\le j-1]}f\left(B^H_{\frac{k-1}{2^n},\frac{j-1}{2^m}}\right)\left[\Delta^{n,m}_{i,j}(B^H)\Delta^{n,m}_{k,l}(B^H) - \frac{1}{4(k-1)(j-1)2^{2nH_1+2mH_2}}\right]$$
$$\xrightarrow{L^2} \int_{R_1\trianglelefteq_1 R_1} f''(B^H_{\mathbf{b}\otimes\mathbf{a}})d\mu_{(H_1,1/2)}(\mathbf{a}) \circ dB^H_{\mathbf{b}}, \quad (6)$$

where $d\mu_{(H_1,1/2)}(\mathbf{a}) = H_1a_1^{2H_1-1}da_1da_2$.

By the direct extension of the case of the Hermite process Z, studied by Nourdin et al. in Nourdin et al. (2010), to the case of Φ^H given below, we prove the asymptotic behaviors of the sequence $\{F_{n,m}\}$ in the case when $(H_1, H_2) \in (1/2, 1) \times (1/2, 1)$. Also by using the direct extension of the case of fBm, studied by Nourdin et al. in Nourdin et al. (2010), to the case of fBs, we study the asymptotic behaviors of the sequence $\{F_{n,m}\}$ in the case when $(H_1, H_2) \in (0, 1/2) \times (0, 1/2)$. Hence, in the case when $(H_1, H_2) \in (0, 1/2) \times (0, 1/2)$ or $(H_1, H_2) \in (1/2, 1) \times (1/2, 1)$, we just describe the results of the limits of the sequence $\{F_{n,m}\}$.