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Convergence rate of maximum likelihood estimator of parameter in stochastic partial differential equation ☆

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ABSTRACT

Using the recent results obtained by combining Malliavin calculus and Stein's method, we study the rate of convergence of the distribution of the maximum likelihood estimator of a parameter appearing in a stochastic partial differential equation. The aim of this paper is to develop the new techniques, allowing us to improve the rate, given by Mishra and Prakasa Rao (2004), to $O(1/N)$.

1. Introduction

We investigate asymptotic properties of the maximum likelihood estimators for parameter $\theta > 0$ occurring in parabolic SPDEs of the following form

$$du(t,x) = \theta\Delta u(t,x) + dW_Q(t,x) \tag{1}$$

with the initial and boundary conditions given by

$$\begin{cases} u(0,x) = f(x), & 0 < x < 1,\ f \in L^2((0,1)) \\ u(t,0) = u(t,1) = 0, & 0 \le t \le T. \end{cases} \tag{2}$$

Here $\Delta = \frac{\partial^2}{\partial x^2}$, and Q is a nuclear covariance operator for the Wiener process $W_Q(t,x)$ taking values in $L^2((0,1))$ so that $W_Q(t,x) = Q^{1/2}W(t,x)$, where W is a cylindrical Brownian motion in $L^2((0,1))$. It is a standard fact (see Rozovskii, 1990) that $W_Q(t,x)$ can be represented as

$$dW_Q(t,x) = \sum_{i=1}^{\infty} \sqrt{q_i} e_i(x) W_i(t) \quad \text{a.s.,}$$

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where $W_i(t)$, $i = 1, 2, \ldots$ are independent one-dimensional Wiener processes and $\{e_i\}$, $i = 1, 2, \ldots,$ is a complete orthonormal system (CONS) in $L^2((0, 1))$ consisting of the eigenvectors of Q, and q_i are the corresponding eigenvalues of Q. In this paper, we consider the special covariance operator Q with CONS $e_i = \sin(i\pi x)$. Then $\{e_i\}$ is a CONS with the eigenvalues $q_i = (1 - \lambda_i)^{-1}$, $i = 1, 2, \ldots,$ for the operator Q, where $\lambda_i = (\pi i)^2$ and $Q = (I - \Delta)^{-1}$ (see Hübner, Khasminskii, & Rozovskii, 1993).

We define a solution $u(t, x)$ of Eq. (1) with initial and boundary conditions (2) as a formal sum

$$u(t, x) = \sum_{i=1}^{\infty} u_i(t)e_i(x),$$

where the Fourier coefficients $u_i(t)$ satisfy the following stochastic differential equation:

$$\begin{cases} du_i(t) = -\theta\lambda_i u_i(t)dt + \dfrac{1}{\sqrt{1+\lambda_i}}dW_i(t), & t > 0, \\ u_i(0) = \nu_i, \quad \nu_i = \displaystyle\int_0^1 f(x)e_i(x)dx. \end{cases} \tag{3}$$

Let $\mathbb{P}_\theta$ be the probability measure generated by the process u on $C([0, T])$. It can be shown that $\mathbb{P}_\theta$ is singular with respect to $\mathbb{P}_{\theta_0}$ when $\theta \neq \theta_0$. Let u^N be the projection of u on $C([0, T])$ onto the subspace spanned by $\{e_1, \ldots, e_N\}$ in $L^2((0, 1))$ and $\mathbb{P}_\theta^N$ be the probability measure corresponding to the process u^N. Then likelihood ratio for the projection u^N can be expressed as

$$\log \frac{d\mathbb{P}_\theta}{d\mathbb{P}_{\theta_0}}(u^N) = -\sum_{i=1}^{N} \lambda_i(\lambda_i + 1)\left\{(\theta - \theta_0)\int_0^T u_i(t)(du_i(t) + \theta_0\lambda_i u_i(t)dt) + \frac{1}{2}(\theta - \theta_0)^2\lambda_i \int_0^T u_i^2(t)dt\right\}. \tag{4}$$

It is clear from (4) that the MLE $\hat{\theta}_N$ of θ based on u^N is given by

$$\hat{\theta}_N = -\frac{\sum_{i=1}^{N} \lambda_i\sqrt{\lambda_i + 1}\int_0^T u_i(t)du_i(t)}{\sum_{i=1}^{N} \lambda_i^2(\lambda_i + 1)\int_0^T u_i^2(t)dt}. \tag{5}$$

In the paper Mishra and Prakasa Rao (2004), the authors obtain a Berry–Esseen type bound for the MLE $\hat{\theta}_N$:

Theorem 1 (*Mishra and Prakasa Rao*)**.** *There exists a constant C depending on θ_0, $\|f\|^2_{L^2((0,1))}$ and T such that for any $\gamma > 0$ and $N \geq N_0$, depending on θ_0 and T,*

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}_{\theta_0}\left(\sqrt{\varphi_N(\theta_0)}(\hat{\theta}_N - \theta_0) \leq z\right) - \mathbb{P}(Z \leq z)\right| \leq CN^{3+\gamma}\left(\frac{1+\sqrt{T}}{TN^3 + \sum_{k=1}^{N} k^4\nu_k^2}\right) + 3\sqrt{N^{-\gamma}}, \tag{6}$$

where the random variable Z has the normal distribution with the zero mean and unit variance, and the normalizing factor $\varphi_N(\theta)$ is

$$\varphi_N(\theta) = \frac{1}{2\theta}\sum_{i=1}^{N} \lambda_i(\lambda_i + 1)\left\{\nu_i^2(1 - e^{-2\theta\lambda_i T}) + \frac{T}{\lambda_i + 1}\right\}.$$

In Remark 4.4 in Mishra and Prakasa Rao (2004), the authors argue that the bound in Theorem 1 is of order $O(N^{\gamma-2}) + O(N^{-\gamma/2})$ provided $\sum_{k=1}^{N} k^4\nu_k^2 \geq g(N) = O(N^5)$, and in such case, the upper bound can be obtained to be of order $O(N^{-2/3})$ by choosing $\gamma = 4/3$. However, if $\sum_{k=1}^{N} k^4\nu_k^2 \leq g(N) = O(N^3)$ (for example, $f = 0$ i.e. $\nu_k = 0$ for all $k = 1, 2, \ldots$), then the upper bound in (6) is given by

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}_{\theta_0}\left(\sqrt{\varphi_N(\theta_0)}(\hat{\theta}_N - \theta_0) \leq z\right) - \mathbb{P}(Z \leq z)\right| \leq CN^{\gamma}. \tag{7}$$

In such case, from the upper bound (7) of the *Kolmogorov distance*, we cannot show that the normal approximation of the MLE $\hat{\theta}_N$ holds. Hence the sharp upper bound is needed to prove the normal approximation through the *Kolmogorov distance*.

Let $\{F_N\}$ be a sequence of zero-mean real-valued random variables with the form of a functional of an infinite dimensional Gaussian fields. In the paper Nourdin and Peccati (2009b), the authors, by combining Malliavin calculus with Stein's method, obtain explicit bounds of the type

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}(F_N \leq z) - \mathbb{P}(Z \leq z)\right| \leq \phi(N), \tag{8}$$