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On the convergence to the multiple subfractional Wiener–Itô integral

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ABSTRACT

In this paper, we construct a family of continuous stochastic processes that converges in law to the multiple Wiener–Itô integrals with respect to the subfractional Brownian motion with $H > \frac{1}{2}$ for the integrand f in a rather general class of functions. We mainly use Donsker and Stroock approximations and the techniques of the multiple Wiener–Itô integral with respect to the Wiener process.

1. Introduction

Recently, self-similarity and long-range dependence properties have become important aspects of stochastic models in various scientific areas including hydrology, telecommunication, turbulence, image processing and finance. The best known and most widely used Gaussian process that exhibits the long-range dependence property is *fractional Brownian motion* (fBm in short). The fBm is a suitable generalization of the standard Brownian motion, but exhibits self-similarity, and it has stationary increments. Another example of a Gaussian process closely related to fBm, but without stationary increments, is the *subfractional Brownian motion* (subfBm for short). The subfBm has properties analogous to those of fBm (self-similarity, long-range dependence, and Hölder paths), enlarges the scope of behavior of fBm and it may be useful in some applications. Bojdecki, Gorostiza, and Talarczyk (2004a) introduced and studied the subfBm, which arises from occupation time fluctuations of branching particle systems with Poisson initial conditions (Bojdecki, Gorostiza, & Talarczyk, 2004b, 2006). More studies on subfBm can be found in Bojdecki, Gorostiza, and Talarczyk (2007), Dzhaparidze and Van Zanten (2004), Liu and Yan (in press), Shen (2011), Shen and Chen (2012), Shen, Chen, and Yan (2011), Shen and Yan (2011), Tudor (2007b), Tudor (2008a), Tudor (2008b), Tudor and Tudor (2006), Yan and Shen (2010) and Yan, Shen, and He (2011) and the references therein.

Let Y be a semimartingale with trajectories belonging to the space $\mathcal{D}([0, T])$ of functions that are right continuous with left limits at all points in $[0, T]$, and define the following iterated Itô integrals

$$J_k(Y)_t = \begin{cases} Y_t & \text{if } k = 1, \\ \displaystyle\int_0^t J_{k-1}(Y)_{s-} dY_s & \text{for } k \geq 2. \end{cases}$$

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Suppose that $\{X^{\epsilon}\}_{\epsilon>0}$ is a family of semimartingales with paths in $\mathcal{D}([0,T])$ that converges weakly in this space to another semimartingale X, as ϵ tends to zero. Avram (1988) proved that in order to obtain the joint weak convergence of $(J_1(X^{\epsilon}),\ldots,J_n(X^{\epsilon}))$ to $(J_1(X),\ldots,J_n(X))$, we need the convergence of X^{ϵ} to X but also the convergence of the quadratic variations. On the other hand, in the literature, when our semimartingale is the Wiener process, there is a lot of important examples of families of processes with absolutely continuous paths converging in law in $\mathcal{C}([0,T])$ to it. In this case, clearly, we do not have convergence of the quadratic variations to that of the Brownian motion. This led to the problem of convergence multiple stochastic integrals with respect to the Brownian motion and later, with respect to the fBm and the subfBm.

Bardina and Jolis (2000) considered the problem of the weak convergence, as ε tends to zero, of the multiple integral processes

$$\left\{\int_0^t\cdots\int_0^t f(t_1,\ldots,t_n)d\eta_{\epsilon}(t_1)\ldots\eta_{\epsilon}(t_n),\, t\in[0,T]\right\}$$

in the space $\mathcal{C}_0([0,T])$, the space of continuous functions defined in $[0,T]$ which are null at zero, where $f\in L^2([0,T]^n)$ is a given function, and $\{\eta_{\epsilon}(t)\}_{\epsilon>0}$ is a family of stochastic processes with absolutely continuous paths that converges weakly in $\mathcal{C}_0([0,T])$ to the Brownian motion, two particular examples of $\eta_{\epsilon}(t)=\int_0^t\theta_{\epsilon}(x)dx$ were considered, i.e., Donsker and Stroock approximations.

The Donsker kernels θ_{ϵ} are the classical kernels appearing in the known Functional Central Limit Theorem,

$$\theta_{\epsilon}(x)=\frac{1}{\epsilon}\sum_{k=1}^{\infty}\zeta_k 1_{[k-1,k)}\left(\frac{x}{\epsilon^2}\right),$$

where ζ_k are independent, centered and identically distributed random variables with $E(\zeta_k^2)=1$.

The Stroock kernels $\theta_{\epsilon}(x)$ were introduced by Stroock (1982) in order to obtain weak approximations of Brownian motion,

$$\theta_{\epsilon}(x)=\frac{1}{\epsilon}(-1)^{N\left(\frac{x}{\epsilon^2}\right)},$$

where $N=\{N(s),s\geq 0\}$ is a standard Poisson process.

We denote by $Y^f_{\eta_{\epsilon}}$ the stochastic processes defined by

$$\begin{aligned}Y^f_{\eta_{\epsilon}}(t)&:=\int_{[0,t]^n}f(x_1,\ldots,x_n)\prod_{\{i,j=1;i\neq j\}}^{n}1_{\{|x_i-x_j|>\epsilon\}}d\eta_{\epsilon}(x_1)\ldots d\eta_{\epsilon}(x_n)\\&=\int_{[0,t]^n}f(x_1,\ldots,x_n)\left(\prod_{i=1}^{n}\theta_{\epsilon}(x_i)\right)\prod_{\{i,j=1;i\neq j\}}^{n}1_{\{|x_i-x_j|>\epsilon\}}dx_1\ldots dx_n,\end{aligned}$$

for all $t\in[0,T]$, $f\in L^2([0,T]^n)$. Bardina, Jolis, and Tudor (2009) studied the weak convergence of the finite dimensional distributions of $(Y^f_{\eta_{\epsilon}})_{\epsilon>0}$ to that of the corresponding multiple Wiener–Itô integral of f with respect to the Wiener process, and also the convergence in $\mathcal{C}_0([0,1])$ of second-order integrals for Donsker and Stroock kernels.

Tudor (2007a) considered that the double Wong–Zakai approximation converges in the L^2 sense and uniformly on every compact time interval, to the double Stratonovich subfractional integral, the integrands are continuous or given by bimeasures. Tudor (2008b) extended the results from Tudor (2007a), for Wong–Zakai and mollifier approximation, to the stronger mean square convergence in the uniform norm. On the other hand, Bardina, Es-Sebaiy, and Tudor (2010) researched approximation of the finite dimensional distributions of multiple fractional integrals.

Inspired by these results, the purpose of this paper is to prove the family $(I_{n_{\epsilon}}(f))_{\epsilon>0}$ defined by

$$\begin{aligned}I_{n_{\epsilon}}(f)_t&:=\int_{[0,1]^n}(\Gamma_n^{H,e}f1_{[0,t]}^{\otimes n})(x_1,\ldots,x_n)\prod_{\{i,j=1;i\neq j\}}^{n}1_{\{|x_i-x_j|>\epsilon\}}d\eta_{\epsilon}(x_1)\ldots\eta_{\epsilon}(x_n)\\&=\int_{[0,1]^n}(\Gamma_n^{H,e}f1_{[0,t]}^{\otimes n})(x_1,\ldots,x_n)\left(\prod_{i=1}^{n}\theta_{\epsilon}(x_i)\right)\prod_{\{i,j=1;i\neq j\}}^{n}1_{\{|x_i-x_j|>\epsilon\}}dx_1\ldots dx_n,\end{aligned}\tag{1.1}$$

converges in law to the multiple Wiener–Itô integrals $I_n^{H,e}(f1_{[0,t]}^{\otimes n})$ with respect to the subfBm, for the integrand f in a rather general class of functions, where $\eta_{\epsilon}(t)=\int_0^t\theta_{\epsilon}(x)dx$ are Donsker and Stroock approximations. Since the expression of the operator $\Gamma_n^{H,e}$ is rather complicate, we cannot apply directly the results in Bardina et al. (2009) to prove our main results. Moreover, the expectation $EI_1^{H,e}(1_A)I_1^{H,e}(1_B)$ is not zero when A and B are disjoint subsets of $[0,1]$ and this fact makes the proofs in this paper more complex than in the standard Brownian motion case.

This paper is organized as follows. In Section 2 we present some preliminaries for subfBm and multiple Wiener–Itô integrals and multiple integrals with respect to the subfBm. In Section 3 we show the convergence results.