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Comparison for upper tail probabilities of random series

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ABSTRACT

Let $\{\xi_n\}$ be a sequence of independent and identically distributed random variables. In this paper we study the comparison for two upper tail probabilities $\mathbb{P}\left\{\sum_{n=1}^{\infty} a_n \mid \xi_n \mid^p \geq r\right\}$ and $\mathbb{P}\left\{\sum_{n=1}^{\infty} b_n \mid \xi_n \mid^p \geq r\right\}$ as $r \to \infty$ with two different real series $\{a_n\}$ and $\{b_n\}$. The first result is for Gaussian random variables $\{\xi_n\}$, and in this case these two probabilities are equivalent after suitable scaling. The second result is for more general random variables, thus a weaker form of equivalence (namely, logarithmic level) is proved.

1. Introduction

Let $\{\xi_n\}$ be a sequence of independent and identically distributed (i.i.d.) random variables, and $\{a_n\}$ be a sequence of positive real numbers. We consider the random series $\sum_{n=1}^{\infty} a_n\xi_n$. Such random series are basic objects in time series analysis and in regression models (see Davis & Resnick, 1991), and there have been a lot of research. For example, Gluskin and Kwapień (1995) and Latala (1996) studied tail probabilities and moment estimates of the random series when $\{\xi_n\}$ have logarithmically concave tails. Of special interest are the series of positive random variables, or the series of the form $\sum_{n=1}^{\infty} a_n|\xi_n|^p$. Indeed, by Karhunen–Loéve expansion, the L_2 norm of a centered continuous Gaussian process $X(t)$, $t \in [0, 1]$, can be represented as $\|X\|_{L_2} = \sum_{n=1}^{\infty} \lambda_n Z_n^2$ where λ_n are the eigenvalues of the associated covariance operator, and Z_n are i.i.d. standard Gaussian random variables. It is also known (see Lifshits, 1994) that the series $\sum_{n=1}^{\infty} a_n|Z_n|^p$ coincides with some bounded Gaussian process $\{Y_t, t \in \mathrm{T}\}$, where T is a suitable parameter set: $\sum_{n=1}^{\infty} a_n|Z_n|^p = \sup_{\mathrm{T}} Y_t$.

In this paper, we study the limiting behavior of the upper tail probability of the series

$$\mathbb{P}\left\{\sum_{n=1}^{\infty} a_n|\xi_n|^p \geq r\right\} \quad \text{as } r \to \infty. \tag{1.1}$$

This probability is also called large deviation probability (see Arcones, 2004). As remarked in Gao and Li (2007), for Gaussian process $\|X\|_{L_2} = \sum_{n=1}^{\infty} \lambda_n Z_n^2$, the eigenvalues λ_n are rarely found exactly. Often, one only knows the asymptotic approximation. Thus, a natural question is to study the relation between the upper tail probability of the original random series and the one with approximated eigenvalues. Also, it is much easier to analyze the rate function in the large deviation theory when $\{a_n\}$ are explicitly given instead of asymptotic approximation.

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Throughout this paper, the following notations will be used. The l^q norm of a real sequence $a = \{a_n\}$ is denoted by $\|a\|_q = \left(\sum_{n=1}^\infty a_n^q\right)^{1/q}$. In particular, the l^∞ norm should be understood as $\|a\|_\infty = \max |a_n|$.

We focus on the following two types of comparisons. The first is at the exact level

$$\frac{\mathbb{P}\left\{\sum_{n=1}^\infty a_n|\xi_n| \geq r\|a\|_2\beta + |\alpha| \sum_{n=1}^\infty a_n\right\}}{\mathbb{P}\left\{\sum_{n=1}^\infty b_n|\xi_n| \geq r\|b\|_2\beta + |\alpha| \sum_{n=1}^\infty b_n\right\}} \sim 1 \quad \text{as } r \to \infty \tag{1.2}$$

where $\{\xi_n\}$ are i.i.d. Gaussian random variables $N(\alpha, \beta^2)$; see Theorems 2.1 and 2.2. This is motivated by Gao, Hannig, and Torcaso (2003) in which the following exact level comparison theorems for small deviations were obtained: as $r \to 0$, $\mathbb{P}\left\{\sum_{n=1}^\infty a_n|\xi_n| \leq r\right\} \sim c\mathbb{P}\left\{\sum_{n=1}^\infty b_n|\xi_n| \leq r\right\}$ for i.i.d. random variables $\{\xi_n\}$ whose common distribution satisfies several weak assumptions in the vicinity of zero. The proof of the small deviation comparison is based on the equivalence form of $\mathbb{P}\left\{\sum_{n=1}^\infty a_n|\xi_n| \leq r\right\}$ introduced in Lifshits (1997). Our proof of upper tail probability comparison (1.2) is also based on an equivalent form of $\mathbb{P}\left\{\sum_{n=1}^\infty a_n|\xi_n| \geq r\right\}$ in Lifshits (1994) for Gaussian random variables. The main difficulty is to come up with suitable inequalities which can be used for a specified function $\widehat{\varepsilon}(x, y)$ in Lemma 2.1, and such inequalities are obtained in Lemmas 2.3 and 2.4.

For more general random variables, difficulties arise due to the lack of known equivalent form of $\mathbb{P}\left\{\sum_{n=1}^\infty a_n|\xi_n| \geq r\right\}$. Thus, instead of exact comparison, we consider logarithmic level comparison for upper tail probabilities

$$\frac{\log \mathbb{P}\left\{\sum_{n=1}^\infty a_n|\xi_n| \geq r\|a\|_q\right\}}{\log \mathbb{P}\left\{\sum_{n=1}^\infty b_n|\xi_n| \geq r\|b\|_q\right\}} \sim 1 \quad \text{as } r \to \infty. \tag{1.3}$$

It turns out that under suitable conditions on the sequences $\{a_n\}$ and $\{b_n\}$ the comparison (1.3) holds true for i.i.d. random variables $\{\xi_n\}$ satisfying

$$\lim_{u\to\infty} u^{-p} \log \mathbb{P}\{|\xi_1| \geq u\} = -c$$

for some finite constants $p \geq 1$ and $c > 0$; see Theorem 3.1. Here we note that logarithmic level comparisons for small deviation probabilities can be found in Gao and Li (2007).

From comparisons (1.2) and (1.3), we see that two upper tail probabilities are equivalent as long as suitable scaling is made. We believe that this holds true for more general random variables; see the conjecture at the end of Section 2 for details.

2. Exact comparisons for Gaussian random series

2.1. The main results

The following two theorems are the main results in this section. The first one is on standard Gaussian random variables.

Theorem 2.1. *Let $\{Z_n\}$ be a sequence of i.i.d. standard Gaussian random variables $N(0, 1)$, and $\{a_n\}$, $\{b_n\}$ be two non-increasing sequences of positive real numbers such that $\sum_{n=1}^\infty a_n < \infty$, $\sum_{n=1}^\infty b_n < \infty$,*

$$\prod_{n=1}^\infty \left(2 - \frac{a_n/\|a\|_2}{b_n/\|b\|_2}\right) \quad \text{and} \quad \prod_{n=1}^\infty \left(2 - \frac{b_n/\|b\|_2}{a_n/\|a\|_2}\right) \text{ converge.} \tag{2.1}$$

Then as $r \to \infty$

$$\mathbb{P}\left\{\sum_{n=1}^\infty a_n|Z_n| \geq r\|a\|_2\right\} \sim \mathbb{P}\left\{\sum_{n=1}^\infty b_n|Z_n| \geq r\|b\|_2\right\}.$$

For general Gaussian random variables Z_n, it turns out that the condition (2.1) is not convenient to derive the comparison because some more complicated terms appear in the proof. Therefore, an equivalent condition in another form is formulated which forms the following comparison.

Theorem 2.2. *Let $\{Z_n\}$ be a sequence of i.i.d. Gaussian random variables $N(\alpha, \beta^2)$, and $\{a_n\}$, $\{b_n\}$ be two non-increasing sequences of positive real numbers such that $\sum_{n=1}^\infty a_n < \infty$, $\sum_{n=1}^\infty b_n < \infty$,*

$$\sum_{n=1}^\infty \left(1 - \frac{a_n/\|a\|_2}{b_n/\|b\|_2}\right) \text{ converges,} \quad \text{and} \quad \sum_{n=1}^\infty \left(1 - \frac{a_n/\|a\|_2}{b_n/\|b\|_2}\right)^2 < \infty. \tag{2.2}$$