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On the sums of probability functions of order statistics[☆]

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ABSTRACT

When a random vector is independent and identically distributed, we have expressed the sums of the marginal probability functions of the order statistics of the random vector in terms of the common marginal probability functions of the random vector. We have also derived the relationships between the sums of the joint probability functions of two order statistics of the random vector and the common marginal probability functions of the random vector.

1. Introduction

Order statistics (Balakrishnan & Cohen, 1991; David & Nagaraja, 2003; Park, Kwon, Kim, & Song, 2006) have attracted much interest in various areas. For example, order statistics have been employed in signal filtering and restoration (Celebi, Schaefer, & Huiyu, 2010), and recently, have been exploited to offer detection performance robust to the inexact statistics of the noise process in the area of wireless communication (He, Jian, Su, Qu, & Gu, 2010; Wang, Jiang, & He, 2009). In addition, order statistics together with rank statistics have been employed in various problems of signal detection (Song, Bae, & Kim, 2002).

It is well-known that the probability functions such as the cumulative distribution function (cdf), probability density function (pdf), and probability mass function (pmf) of the order statistics of an independent and identically distributed (i.i.d.) random vector can be obtained once the probability functions of the random vector are specified.

In this paper, we derive expressions of the sums of the cdfs, pdfs, and pmfs of the order statistics of an i.i.d. random vector in terms of the common marginal cdf, pdf, and pmf, respectively, of the random vector.

2. Preliminary

Consider an i.i.d. random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ with n elements. Let the common (marginal) cdf of $\{X_i\}_{i=1}^n$ be F, and let us denote the order statistics of $\underline{X}$ by $\left\{X_{(r)}\right\}_{r=1}^n$, where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$. The cdf of the r-th order statistic

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$X_{(r)}$ is denoted as $F_{X_{(r)}}$. In the continuous domain, the pdf of $X_{(r)}$ is denoted as $f_{X_{(r)}}$. Similarly, the pmf of $X_{(r)}$ is denoted as $p_{X_{(r)}}$ in the discrete case, in which we assume that the support of the cdf and pmf is the set $\{0, 1, \ldots\}$ of non-negative integers.

As it is well-known (David & Nagaraja, 2003; Song et al., 2002), the cdf $F_{X_{(r)}}$, pdf $f_{X_{(r)}}$, and pmf $p_{X_{(r)}}$ of the r-th order statistic $X_{(r)}$ for $r = 1, 2, \ldots, n$ can be obtained as

$$F_{X_{(r)}}(x) = \sum_{j=r}^{n} \binom{n}{j} F^j(x)\{1 - F(x)\}^{n-j}, \tag{1}$$

$$f_{X_{(r)}}(x) = n\binom{n-1}{r-1} F^{r-1}(x)\{1 - F(x)\}^{n-r} f(x), \tag{2}$$

and

$$p_{X_{(r)}}(x) = n\binom{n-1}{r-1} \int_{F(x-1)}^{F(x)} v^{r-1}(1-v)^{n-r} dv \tag{3}$$

from the common cdf F, where

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} \tag{4}$$

is the binomial coefficient and

$$f(x) = \frac{d}{dx}F(x) \tag{5}$$

is the common pdf of $\{X_i\}_{i=1}^n$ in the case of continuous random vectors.

Next, denote by $F_{X_{(r)},X_{(s)}}$ the joint cdf of the r-th order statistic $X_{(r)}$ and the s-th order statistic $X_{(s)}$. The joint pdf $f_{X_{(r)},X_{(s)}}$ and joint pmf $p_{X_{(r)},X_{(s)}}$ are similarly defined. Then, the joint cdf $F_{X_{(r)},X_{(s)}}$ for $r < s$ and $r, s \in \{1, 2, \ldots, n\}$ is expressed as (David & Nagaraja, 2003)

$$F_{X_{(r)},X_{(s)}}(x, y) = \begin{cases} \displaystyle\sum_{j=s}^{n} \sum_{i=r}^{j} \binom{n}{j}\binom{j}{i} F^i(x)\{F(y) - F(x)\}^{j-i} \cdot \{1 - F(y)\}^{n-j}, & x \le y, \\ F_{X_{(s)}}(y), & x \ge y. \end{cases} \tag{6}$$

Similarly, for $r < s$ and $r, s \in \{1, 2, \ldots, n\}$, the joint pdf $f_{X_{(r)},X_{(s)}}$ and joint pmf $p_{X_{(r)},X_{(s)}}$ can be obtained as (David & Nagaraja, 2003)

$$f_{X_{(r)},X_{(s)}}(x, y) = \begin{cases} n(n-1)\dbinom{n-2}{s-2}\dbinom{s-2}{r-1} F^{r-1}(x) \cdot \{F(y) - F(x)\}^{s-r-1} \cdot \{1 - F(y)\}^{n-s} f(x) f(y), & x \le y, \\ 0, & x > y, \end{cases} \tag{7}$$

and

$$p_{X_{(r)},X_{(s)}}(x, y) = \begin{cases} n(n-1)\dbinom{n-2}{s-2}\dbinom{s-2}{r-1} \displaystyle\iint_{D_0} v^{r-1} \cdot (w - v)^{s-r-1}(1 - w)^{n-s} dv dw, & x \le y, \\ 0, & x > y, \end{cases} \tag{8}$$

where

$$D_0 = \{(v, w) : F(x-1) \le v \le F(x),\ F(y-1) \le w \le F(y),\ v \le w\} \tag{9}$$

denotes the range of integration.

3. Main results

In this section, for an i.i.d. random vector $\underline{X}$ with common marginal cdf F, we derive the relationships between the sums of the cdfs, pdfs, and pmfs of the order statistics of $\underline{X}$ and the cdf F.