Journal of the Korean Statistical Society 41 (2012) 177–187

Contents lists available at SciVerse ScienceDirect

Journal of the Korean Statistical Society

journal homepage: www.elsevier.com/locate/jkss

# Remarks on asymptotic behavior of weighted quadratic variation of subfractional Brownian motion[☆]

Junfeng Liu[a,*], Litan Yan[b]

[a] Department of Mathematics, Nanjing Audit University, 86 West Yushan Rd., Pukou, Nanjing 211815, PR China

[b] Department of Mathematics, Donghua University, 2999 North Renmin Rd., Songjiang, Shanghai 201620, PR China

ARTICLE INFO

*Article history:*
Received 19 March 2011
Accepted 11 August 2011
Available online 1 September 2011

*AMS subject classifications:*
60H07
60G15
60G22

*Keywords:*
Subfractional Brownian motion
Malliavin calculus
Weighted quadratic variation

ABSTRACT

The present note is devoted to prove, by means of Malliavin calculus, the convergence in $L^2$ of some properly renormalized weighted quadratic variation of sub-fractional Brownian motion $S^H$ with parameter $H < \frac{1}{4}$.



## 1. Introduction

The characterization of single path behavior of a given stochastic process is often based on the study of its power variations. A quite extensive literature on the subject has been developed on this subject; see e.g. Corcuera, Nualart, and Woerner (2006) and Gradinaru and Nourdin (2009) for references concerning the power variations of Gaussian and Gaussian-related processes, and Barndorff-Nielsen, Graversen, and Shepard (2004) (and the references therein) for applications of power variation techniques to the continuous time modeling of financial markets. Recall that, a real $p > 1$ being given, the $p$-power variation of a process $X$, with respect to a subdivision $\pi_n = \left\{0 = t_{n,0} < t_{n,1} < \cdots < t_{n,p(n)} = 1\right\}$ of $[0, 1]$, is defined to be the sum

$$\sum_{k=1}^{p(n)} \left|X_{t_{n,k}} - X_{t_{n,k-1}}\right|^p . \tag{1.1}$$

For simplicity, consider from now on the case where $t_{n,k} = k/n$, for $n \in \mathbb{N}^*$ and $k \in \{0, 1, 2, \dots, n\}$. When weights are introduced in (1.1), some interesting phenomenon appears. More precisely, consider quantities such as

$$\sum_{k=0}^{n-1} h\left(X_{k/n}\right) \left(\Delta X_{k/n}\right)^p , \tag{1.2}$$

☆ This project was sponsored by NSFC (10871041) and NSRC (10023).

* Corresponding author.
*E-mail address:* jordanjunfeng@163.com (J. Liu).

1226-3192/$ – see front matter 
doi:10.1016/j.jkss.2011.08.002

where the function $h : \mathbb{R} \to \mathbb{R}$ is assumed to be smooth enough and $\Delta X_{k/n} = X_{(k+1)/n} - X_{k/n}$. Notice (1.2) is called weighted power variations because of the presence of the factor $h(X_{k/n})$.

Very recently, for $X = B^H$, the fractional Brownian motion with Hurst index $H \in (0, 1)$, the asymptotic behavior of

$$\sum_{k=0}^{n-1} h\left(B^H_{k/n}\right)\left[n^{2H}\left(\Delta B^H_{k/n}\right)^2 - 1\right], \tag{1.3}$$

received a lot of attentions (see Gradinaru & Nourdin, 2009; Neuenkirch & Nourdin, 2007). The analysis of the asymptotic behavior of quantities of type (1.3) is motivated, for instance, by the study of the exact rates of convergence of some approximation schemes of scalar stochastic differential equations driven by $B^H$ (see Gradinaru & Nourdin, 2009; Neuenkirch & Nourdin, 2007 and references therein for precise statements), besides, of course, the traditional applications of quadratic variations to parameter estimation problems. But it turned out that it was also interesting because it highlighted new phenomena with respect to some classical results obtained in the seminal works by Breuer and Major (1983), Dobrushin and Major (1979), Giraitis and Surgailis (1985) or Taqqu (1979). Indeed, we know that, for any $0 < H < 3/4$, the convergence

$$\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\left[n^{2H}\left(\Delta B^H_{l/n}\right)^2 - 1\right] \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0, \sigma_H^2), \tag{1.4}$$

holds, where $\sigma_H$ denotes a constant depending only $H$ which can be computed explicitly.

Nourdin (2008) began the study of asymptotic analysis of (1.3). If $H < 1/4$, he proved that

$$n^{2H-1}\sum_{l=0}^{n-1} h\left(B^H_{l/n}\right)\left[n^{2H}\left(\Delta B^H_{l/n}\right)^2 - 1\right] \xrightarrow[n\to\infty]{L^2} \frac{1}{4}\int_0^1 h''(B^H_s)\mathrm{d}s, \tag{1.5}$$

holds.

As pointed out by Nourdin (2008), (1.5) is somewhat surprising when compared with (1.4). Indeed, instead of an $L^2$-convergence, we only have a convergence in law in (1.4). Observe that, since $2H - 1 < 1/2$ if and only if $H < 1/4$, convergence (1.4) and (1.5) are, of course, not contradictory.

Furthermore, the results of Nourdin (2008) have been improved by Nourdin and Réveillac (2009) and Nourdin, Nualart, and Tudor (2010). More precisely: If $H = \frac{1}{4}$, then

$$\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} h\left(B^H_{l/n}\right)\left[\sqrt{n}\left(\Delta B^{1/4}_{l/n}\right)^2 - 1\right] \xrightarrow[n\to\infty]{\text{Law}} C_{1/4}\int_0^1 h(B^{1/4}_s)\mathrm{d}W_s + \frac{1}{4}\int_0^1 h''(B^{1/4}_s)\mathrm{d}s, \tag{1.6}$$

for $W$ a standard Brownian motion independent of $B^{1/4}$ and where

$$C_{1/4} = \sqrt{\frac{1}{2}\sum_{p=-\infty}^{\infty}\left(\sqrt{|p+1|} + \sqrt{|p-1|} - 2\sqrt{|p|}\right)^2} \approx 1535.$$

If $\frac{1}{4} < H < \frac{3}{4}$, then

$$\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} h\left(B^H_{l/n}\right)\left[n^{2H}\left(\Delta B^H_{l/n}\right)^2 - 1\right] \xrightarrow[n\to\infty]{\text{Law}} C_H\int_0^1 h(B^H_s)\mathrm{d}W_s, \tag{1.7}$$

for $W$ a standard Brownian motion independent of $B^H$.

If $H = \frac{3}{4}$, then

$$\frac{1}{\sqrt{n\log n}}\sum_{l=0}^{n-1} h\left(B^{3/4}_{l/n}\right)\left[n^{3/2}\left(\Delta B^{3/4}_{l/n}\right)^2 - 1\right] \xrightarrow[n\to\infty]{\text{Law}} C_{3/4}\int_0^1 h(B^{3/4}_s)\mathrm{d}W_s, \tag{1.8}$$

for $W$ a standard Brownian motion independent of $B^{3/4}$;

If $H > \frac{3}{4}$, then

$$n^{1-2H}\sum_{l=0}^{n-1} h\left(B^H_{l/n}\right)\left[n^{2H}\left(\Delta B^H_{l/n}\right)^2 - 1\right] \xrightarrow[n\to\infty]{L^2} \int_0^1 h(B^H_s)\mathrm{d}Z_s, \tag{1.9}$$

for $Z$ the Rosenblatt process defined by

$$Z_s = I_2^X(L_s), \tag{1.10}$$