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Almost sure limit theorem for stationary Gaussian random fields

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ABSTRACT

We obtain an almost sure version of a maximum limit theorem for stationary Gaussian random fields under some covariance conditions. As a by-product, we also obtain a weak convergence of the stationary Gaussian random field maximum, which is interesting independently.

1. Introduction

Random field theory has recently received increasing attention. Applications of random field theory are extremely numerous and diverse. These include image analysis, atmospheric sciences and geostatistics, among others. In particular, Gaussian random fields are an important class for several reasons: Many natural phenomena are reasonably modelled by them, the model is specified by expectations and covariances, and its estimation and inference are relatively simple. The book by Adler and Taylor (2007) is a good reference for general random field theory including Gaussian random fields and Piterbarg (1996) deals with the maximum of Gaussian random fields in detail.

In this paper we consider the maximum of stationary Gaussian random fields with discrete parameter and obtain its almost sure limit theorem. The almost sure version of the central limit theorem was first discovered by Brosamler (1988) and Schatte (1988), and has been discussed by various authors. Its general form is that if the partial sums $S_n = \sum_{k=1}^n \xi_k$ for a sequence of random variables $\xi_1, \xi_2, \ldots$ satisfy $a_n(S_n - b_n) \overset{\mathcal{D}}{\to} G$ for some constants $\{a_n\}, \{b_n\}$ and a distribution function G, then under some conditions

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I(a_k(S_k - b_k) \le x) = G(x) \quad \text{a.s.}$$

for any continuity point x of G, where $I(A)$ denotes the indicator function for the event A. Recently Cheng, Peng, and Qi (1998) and Fahrner and Stadmüller (1998) extended this principle by establishing the almost sure limit theorem for the maximum of independent identically distributed random variables. In particular, Csáki and Gonchigdanzan (2002) and Shouquan and Zhengyan (2007) obtained the almost sure limit theorems for the maxima of stationary Gaussian sequences. The weak convergence of the linearly transformed maximum of a Gaussian sequence is well known, e.g. if M_n is the maximum of

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a (standard) Gaussian sequence $\{\xi_n, n \geq 1\}$ of random variables, i.e. $M_n = \max_{i \leq n} \xi_i$, we have under appropriate covariance conditions

$$P\{a_n(M_n - b_n) \leq x\} \rightarrow \exp(-e^{-x}) \quad \text{for every } x \in \mathbb{R}, \tag{1}$$

where $a_n = \sqrt{2 \log n}$ and $b_n = a_n - \frac{\log \log n + \log 4\pi}{2a_n}$. Csáki and Gonchigdanzan (2002) proved

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I(a_k(M_k - b_k) \leq x) = \exp(-e^{-x}) \quad \text{a.s. for every } x \in \mathbb{R} \tag{2}$$

with the same $\{a_n\}$, $\{b_n\}$ as in (1), by imposing additional conditions on the covariances, which conditions are weakened by Shouquan and Zhengyan (2007). The present work is devoted to the study of the stationary Gaussian random field aspect of (2) under the modified version of the covariance condition assumed by Shouquan and Zhengyan (2007). As a by-product, we obtain the weak convergence result for the maximum of stationary Gaussian random fields (see Lemma 3).

In the sequel, $\{X_j\}$ indexed by $j \equiv (j_1, \ldots, j_d)$ in $\mathbb{N}^d$ is assumed to be stationary standardized Gaussian random fields with covariance $r_n = \mathrm{Cov}(X_j, X_{j+n})$, $n \in \mathbb{Z}^d$, where $\mathbb{N}$ and $\mathbb{Z}$ denote the set of positive integers and the set of integers respectively. We assume that the sampling region is given by $J_n \equiv \{j \in \mathbb{N}^d : 1 \leq j_i \leq n_i,\ i = 1, \ldots, d\}$, where $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$. Let $M_n = \max_{j \in J_n} X_j$ and if E is any subset of $\mathbb{N}^d$, $M(E)$ will denote $\max_{j\in E} X_j$ and, of course, $M_n = M(J_n)$.

The rest of the paper is organized as follows. Section 2 contains the statement of the main result and its proof, which is achieved by a series of preliminary lemmas collected in Appendix.

In the sequel, λ_E is the number of elements in E for any subset E of $\mathbb{N}^d$. Let $\lambda_k = \prod_{i:k_i \neq 0} |k_i|$ for $k = (k_1, \ldots, k_d)$ and $\lambda_0 = 1$. Again, of course, $\lambda_k = \lambda_{J_k}$ when $k \in \mathbb{N}^d$. Also let $\log n$ denote $(\log n_1, \ldots, \log n_d)$. We express the condition that $\min_{1\leq i\leq d} n_i \rightarrow \infty$ and $n_j/n_i < K$ for some $K > 0$, $1 \leq i, j \leq d$ by simply writing $n \rightarrow \infty$. Let $\Phi(\cdot)$ and $\phi(\cdot)$ denote the standard Gaussian distribution function and its density function respectively. Also for brevity, let $I_n = \{j \in \mathbb{Z}^d : -n_i \leq j_i \leq n_i, i = 1, \ldots, d, j \neq 0\}$. In the case that $\{X_j\}$ is isotropic, the set I_n may be replaced by J_n.

2. Main result

We obtain the following almost sure limit results for the maxima of the stationary Gaussian random fields $\{X_j\}$. Throughout this paper, we use K for a positive constant whose value may change from line to line and let ax, $a \in \{-1, 1\}^d$ for any d-dimensional real vector x denote $(a_1x_1, \ldots, a_dx_d)$.

Theorem 1. *Suppose $r_{an} \rightarrow 0$ for any $a \in \{-1, 1\}^d$ as $n \rightarrow \infty$, satisfying*

$$\lambda_n^{-1} \sum_{k\in I_n} |r_k| \log \lambda_k \exp\{\gamma |r_k| \log \lambda_k\} = O((\lambda_{\log n})^{-\epsilon}) \tag{3}$$

for some $\epsilon > 0$, $\gamma > 2d$.

(i) *If $\{u_n\}$ is constants such that $\lambda_n(1 - \Phi(u_n)) \rightarrow \tau$ for $0 \leq \tau < \infty$ and $u_i < u_j$ when $i < j$, then*

$$\lim_{n\to\infty} \frac{1}{\lambda_{\log n}} \sum_{k\in J_n} \frac{1}{\lambda_k} I(M_k \leq u_k) = \exp(-\tau) \quad a.s.$$

(ii) *If $a_n = \sqrt{2 \log \lambda_n}$ and $b_n = a_n - \frac{\log \log \lambda_n + \log(4\pi)}{2a_n}$, then*

$$\lim_{n\to\infty} \frac{1}{\lambda_{\log n}} \sum_{k\in J_n} \frac{1}{\lambda_k} I(a_k(M_k - b_k) \leq x) = \exp(-e^{-x}) \quad a.s. \text{ for every } x \in \mathbb{R}.$$

The above result still holds under the condition $\lambda_n^{-1} \sum_{k\in I_n} |r_k| \log \lambda_k = O((\lambda_{\log n})^{-\epsilon})$ instead of (3), which is slightly stronger. When $\{X_j\}$ is isotropic, $r_{an} = r_n$ for any $a \in \{-1, 1\}^d$ and hence the condition $r_{an} \rightarrow 0$ is equivalent to the condition $r_n \rightarrow 0$.

Proof of Theorem 1. We first note that the condition (3) implies

$$\lambda_n^{-1} \sum_{k\in I_n} |r_k| \log \lambda_k \exp\{\gamma |r_k| \log \lambda_k\} \rightarrow 0 \tag{4}$$

and hence $P(M_n \leq u_n) \rightarrow e^{-\tau}$ under the conditions of the theorem according to Lemma 3(i) in Appendix. Then we have the partial sums of $P(M_k \leq u_k)$ with weights $\frac{1}{\lambda_{\log n}\lambda_k}$ converge, i.e.

$$\lim_{n\to\infty} \frac{1}{\lambda_{\log n}} \sum_{k\in J_n} \frac{1}{\lambda_k} P(M_k \leq u_k) = \exp(-\tau).$$