Journal of the Korean Statistical Society 39 (2010) 523–531

Contents lists available at ScienceDirect

Journal of the Korean Statistical Society

journal homepage: www.elsevier.com/locate/jkss

Wick integration with respect to fractional Brownian sheet☆

Yoon Tae Kim *

Department of Statistics, Hallym University, Chuncheon 200-702, Republic of South Korea

ARTICLE INFO

Article history:
Received 17 September 2009
Accepted 2 December 2009
Available online 21 December 2009

AMS 2000 subject classifications:
primary 60H07
secondary 60G18

Keywords:
Fractional Brownian sheet
Itô formula
White noise theory
Wick integral

ABSTRACT

By using the white noise theory for a fractional Brownian sheet, we find sufficient conditions on the integrability of the Wick integrals of various types with respect to a fractional Brownian sheet with Hurst parameters $H_1, H_2 \in (0, 1)$. Also we show that the Wick integrals can be represented as formal expansions.

1. Introduction and notations

For $H_1, H_2 \in (0, 1)$, a real-valued fractional Brownian sheet (fBs) B^H with Hurst parameter $H = (H_1, H_2)$ is a centered Gaussian random field with covariance

$$\mathbb{E}[B^H(a)B^H(b)] = \prod_{i=1}^{2} \frac{1}{2} \left(|a_i|^{2H_i} + |b_i|^{2H_i} - |a_i - b_i|^{2H_i}\right),$$

where $a = (a_1, a_2), b = (b_1, b_2) \in R^2$. The theory of stochastic calculus for fBs has recently been developed by several authors (see Kim, 2008; Kim & Jeon, 2006; Kim, Jeon, & Park, 2008, 2009; Kim & Park, 2009; Tudor & Viens, 2003). In particular, by using Malliavin calculus, various types of stochastic integrals with respect to fBs are introduced in Kim et al. (2008). On the other hand, by using the white noise theory, Kim and Jeon (2006) define the following stochastic integrals with respect to fBs: for $z = (z_1, z_2) \in R^2$

$$\int_{R_z} \alpha(a)\mathrm{d}B^H(a), \qquad \int_{R_z \trianglelefteq_1 R_z} \beta(a, b)\mathrm{d}B^H(a)\mathrm{d}B^H(b),$$
$$\int_{R_z \trianglelefteq_1 R_z} \beta(a, b)\mathrm{d}a\mathrm{d}B^H(b), \qquad \int_{R_z \trianglelefteq_1 R_z} \beta(a, b)\mathrm{d}B^H(a)\mathrm{d}b, \tag{1}$$

where $R_z = (0, z_1] \times (0, z_2]$ and the set $R_z \trianglelefteq_1 R_z$ will be defined below. Also Kim and Jeon (2006) establish an Itô formula for fBs by using the above integrals.

☆ This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2009-0073007).

* Tel.: +82 33 248 2035; fax: +82 33 240 1401.
E-mail address: ytkim@hallym.ac.kr.

1226-3192/$ – see front matter © 2010 Published by Elsevier B.V. on behalf of The Korean Statistical Society.
doi:10.1016/j.jkss.2009.12.001

In this paper, by using the white noise calculus for fBs, we find sufficient conditions on the integrability of the Wick integrals given in (1) in the case when the integrands belong to the space of stochastic distributions and admit formal expansions. Furthermore, we will show that the Wick integrals given in (1) can be represented as formal expansions. Also a new representation of the double stochastic integral in (1) will be considered. The integrator of this integral is induced by the Wick product of two partial fractional white noises.

We give some notations that are used throughout the paper. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be two points in R^2.

- The notation $a \preceq b$ will denote the condition $a_1 \le b_1$ and $a_2 \le b_2$.
- The notation $a \trianglelefteq_1 b$ will denote the condition $a_1 \le b_1$ and $a_2 \ge b_2$.
- The notation $a * b$ will denote the point (a_1, b_2).
- For $a \preceq b$, the notation $R_{[a,b]}$ will denote the rectangle $[a, b] = [a_1, b_1] \times [a_2, b_2]$ and $R_{[0,b]} = R_b$.
- For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$ and $\mathrm{d}x^\alpha = \prod_{i=1}^n \mathrm{d}x_i^{\alpha_i}$.
- The notation $R_z \trianglelefteq_1 R_z$ will denote the set $\{(a, b) \in R_z \times R_z : b \trianglelefteq_1 a\}$.

2. Preliminaries

In this section we briefly mention the white noise theory, given in Biagini, Øksendal, Sulem, and Wallner (2004), Elliott and Van der Hoek (2003), Holden, ØKsendal, Uboe, and Zhang (1996), Hu, ØKsendal, and Zhang (2004), and Mishura (2008), to be necessary for our works. Let $\mathcal{S}(R^2)$ be the Schwartz space of rapidly decreasing smooth functions on R^2. We consider the white noise space $(\Omega, \mathbf{F}, \mathbb{P})$ as the underlying probability space, i.e., $\Omega := \mathcal{S}'(R^2)$ is the space of tempered distributions and $\mathbb{P}$ is an unique probability measure such that

$$\int_{S'(R^2)} \mathrm{e}^{\mathrm{i}\langle\omega,f\rangle}\mathrm{d}\mathbb{P}(\omega) = \mathrm{e}^{-(1/2)\|f\|^2_{L^2(R^2)}} \quad \text{for } f \in \mathcal{S}(R^2).$$

Then we have the isometry $\mathbb{E}[\langle\cdot, f\rangle\langle\cdot, g\rangle] = (f, g)_{L^2(R^2)}$, and using this we can extend $\langle\cdot, f\rangle$ to $f \in L^2(R^2)$. For $a, b \in R^2$, we define $\mathbf{1}_{(a,b)}(x) = \prod_{i=1}^2 \mathbf{1}_{(a_i,b_i)}(x_i)$ for $x = (x_1, x_2)$, where the indicator function $\mathbf{1}_{(a_i,b_i)}(x_i)$ is given by

$$\mathbf{1}_{(a_i,b_i)}(x_i) = \begin{cases} 1 & \text{for } a_i \le x_i \le b_i \\ -1 & \text{for } b_i \le x_i \le a_i \\ 0 & \text{otherwise.} \end{cases}$$

For $f \in \mathcal{S}(R^2)$, we define an operator $I_{H_i}f : R^2 \to R$, $i = 1, 2$, by

$$I_{H_i}f(x) = \begin{cases} C_{H_i}\displaystyle\int_R \frac{f(x+u\epsilon_i)}{|u|^{(3/2)-H_i}}\mathrm{d}u & \text{for } 1/2 < H_i < 1 \\ f & \text{for } H_i = \dfrac{1}{2} \\ C_{H_i}\displaystyle\int_R \frac{f(x-u\epsilon_i)-f(x)}{|u|^{(3/2)-H_i}}\mathrm{d}u & \text{for } 0 < H_i < 1/2, \end{cases} \tag{2}$$

where $\epsilon_1 = (1, 0)$, $\epsilon_2 = (0, 1)$ and

$$C_{H_i} = \frac{\sin(\pi H_i)\Gamma(2H_i+1)}{2\Gamma(H_i-(1/2))\cos(\pi/2)(H_i-(1/2))}.$$

Let $I_Hf(x) = I_{H_1}(I_{H_2})f(x)$. Then a continuous version of $\langle\cdot, I_H\mathbf{1}_{(0,a)}\rangle$ is fBs with arbitrary Hurst parameters $H = (H_1, H_2)$, $H_1, H_2 \in (0, 1)$, on $(\Omega, \mathbf{F}, \mathbb{P})$.

Let $\mathbf{H}_n(x)$ and h_n, $n = 0, 1, \ldots$, be the nth Hermite polynomial and the nth Hermite function respectively. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ (with $\mathbb{N} = \{1, 2, \ldots\}$), let us set $\mathbf{e}_\alpha(x_1, x_2) = \prod_{i=1}^2 h_{\alpha_i}(x_i)$. Denote by $\mathbf{A}$ the set of all finite sequences $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ with $a_i \in \{0\} \cup \mathbb{N}$, $m = 1, 2, \ldots$. For $\mathbf{a} \in \mathbf{A}$, we set $\mathbf{a}! = \prod_{i=1}^\infty a_i!$ and $|\mathbf{a}| = \sum_{i=1}^\infty a_i$. Let $\alpha^{(i)}$, $i = 1, 2, \ldots$, be a fixed ordering of $\mathbb{N}^2$ such that for $i < j$, $|\alpha^{(i)}| \le |\alpha^{(j)}|$, which implies that for some constant $C > 0$, $|\alpha^{(k)}| \le Ck$ for all $k \in \mathbb{N}$. With these notations, we define

$$\mathbf{H}_\mathbf{a}(\omega) = \prod_{i=1}^\infty \mathbf{H}_{a_i}(\langle\omega, \mathbf{e}_{\alpha^{(i)}}\rangle).$$

We recall the following chaos expansion theorem.

Theorem 1. *Let $F \in \mathbb{L}^2 := L^2(\Omega, \mathbf{F}, \mathbb{P})$. Then there exist constants $c_\mathbf{a} \in \mathbb{R}$ for $\mathbf{a} \in \mathbf{A}$ such that*

$$F(\omega) = \sum_{\mathbf{a}\in\mathbf{A}} c_\mathbf{a}\mathbf{H}_\mathbf{a}(\omega) \quad \text{limit in } \mathbb{L}^2, \tag{3}$$

where "limit in $\mathbb{L}^2$" means that the infinite series converges in $\mathbb{L}^2$-sense. Furthermore, we have the isometry $\|F\|^2_{\mathbb{L}^2} = \sum_{\mathbf{a}\in\mathbf{A}} \mathbf{a}!c^2_\mathbf{a}$.