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Fractional integrated GARCH diffusion limit models[☆]

T. Plienpanich[a], P. Sattayatham[a,*], T.H. Thao[b]

[a] School of Mathematics, Institute of Science, Suranaree University of Technology, Nakhon Ratchasima, Thailand

[b] Institute of Mathematics, Vietnam Academy of Science and Technology, Viet Nam

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ABSTRACT

In this paper, we introduce an approximate approach to the fractional integrated GARCH(1,1) model of continuous time perturbed by fractional noise. Based on the L^2-approximation of this noise by semimartingales, we proved a convergence theorem concerning an approximate solution. A simulation example shows a significant reduction of error in a fractional stock price model as compared to the classical stock price model.

1. Introduction

Risk in a financial market is measured by using volatility. So predictability of volatility has important implications for risk management. If volatility increases, so will Value At Risk (VAR). Investors may want to adjust their portfolio to reduce their exposure to those assets whose volatility is predicted to increase. One method that is widely employed for volatility estimation is to use GARCH models. A discrete time GARCH(1,1) model is a model of the form

$$\nu_{k+1} = \omega_0 + \beta\nu_{k+1} + \alpha\nu_k U_k^2, \qquad X_k = \sigma_k U_k \tag{1}$$

where $\sigma_k = \sqrt{\nu_k}$, and α, β are weight parameters, ω_0 is a parameter related to the long-term variance, and U_K is a sequence of independent normal random variables with zero mean and variance of 1.

It is well known that GARCH models are not designed for long range-dependence (LRD). But there are some empirical studies showing that log-return series (X_t) of foreign exchange rates, stock indices and share prices exhibit the LRD effect (see, for example, Mikosch and Starica (2003, page 445)). In 1990, Nelson (1990) showed that as the time interval decreases and become infinitesimal, Eq. (1) can be changed to

$$\mathrm{d}v_t = (\omega - \theta v_t)\mathrm{d}t + \xi v_t \mathrm{d}W_t \tag{2}$$

where $v_t = \sigma_t^2$ is the stock-return variance, ω, θ and ξ are weight parameters and W_t is a standard Brownian motion process. To be more accurate, there is a weak convergence of the discrete GARCH process to the continuous diffusion limit. The purpose of this paper is to introduce LRD effect into GARCH models of continuous time (i.e., into Eq. (2)). The importance of this process in finance is that it can be used to forecast volatility and risk of some financial instruments.

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[*] Corresponding author. Fax: +66 44224315.
E-mail address: pairote@sut.ac.th (P. Sattayatham).

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Recall that a fractional Brownian motion process W_t^H, with Hurst index H, is a centered Gaussian process such that its covariance function $R(t,s) = EW_t^H W_s^H$ is given by

$$R(s,t) = \frac{1}{2}(|t|^\gamma + |s|^\gamma - |t-s|^\gamma)$$

where $\gamma = 2H$ and $0 < H < 1$. If $H = \frac{1}{2}$, then W_t^H is the usual standard Brownian motion process. For $H \neq \frac{1}{2}$, W_t^H is neither a semimartingale nor a Markov process so we cannot apply the standard stochastic calculus for this process. It is a process of long range dependence in the following sense: If $\rho_n = E[W_1^H(W_{n+1}^H) - W_n^H]$, then the series $\sum_{n=0}^{\infty} \rho_n$ is either divergent or convergent with very late rate. It is known that a fractional Brownian motion W_t^H can be decomposed as follows:

$$W_t^H = \frac{1}{\Gamma(1+\alpha)}\left[Z_t + \int_0^t (t-s)^\alpha \mathrm{d}W_s\right],$$

where Γ is the gamma function, $Z_t = \int_{-\infty}^0 [(t-s)^\alpha - (-s)^\alpha]\mathrm{d}W_s$, $\alpha = H - \frac{1}{2}$, and W_t is a standard Brownian motion.

We suppose from now on $0 < \alpha < \frac{1}{2}$ so that $\frac{1}{2} < H < 1$. Then Z_t has absolutely continuous trajectories and it is the term $B_t^H := \int_0^t (t-s)^\alpha \mathrm{d}W_s$ that exhibits long range dependence. We will use B_t^H instead of W_t^H in fractional stochastic calculus.

In Thao (2006) constructed an approximate process B_t^ε of B_t^H as follows:

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} \mathrm{d}W_s$$

where $\frac{1}{2} < H < 1$, and W_t is a standard Brownian motion. He also proved that $B_t^\varepsilon \to B_t^H$ in $L^2(\Omega)$ as $\varepsilon \to 0$ (uniformly in t) and B_t^ε is a semimartingale. These results give us a convenient way to study fractional Brownian motions since we can use the standard Ito integrals and then it is easy to do numerical approximation.

By a fractional integrated GARCH model of continuous time (FIGARCH), we shall mean a process of the form

$$\mathrm{d}v_t = (\omega - \theta v_t)\mathrm{d}t + \xi v_t \mathrm{d}B_t^H \tag{3}$$

where $0 \le t \le T$, ω, θ and ξ are weight parameters, and B_t^H is a fractional Brownian motion. For each $\epsilon > 0$, an approximate model of the FIGARCH model is a process of the form

$$\mathrm{d}v_t^\varepsilon = (\omega - \theta v_t^\varepsilon)\mathrm{d}t + \xi v_t^\varepsilon \mathrm{d}B_t^\varepsilon \tag{4}$$

where B_t^ε is the approximate process of B_t^H. We shall show that its solution converges to the solution of the FIGARCH model (3).

Moreover, geometric Brownian motion for the asset price was used to simulate the SCB stock prices where the volatility of this model was predicted from an approximate fractional variance process of GARCH(1,1) model in continuous time and classical GARCH(1,1) model in continuous time. And both of them were compared with the empirical historical stock prices of SCB.

2. Solutions of the approximate models

In this section, we are interested in finding a solution of the approximate model (4) together with initial condition $v_{t(t=0)}^\varepsilon = v_0$.

Let $\varepsilon > 0$. Recall that an approximated process B_t^ε is defined by

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha \mathrm{d}W_s,$$

where $\alpha = H - \frac{1}{2}$, $0 < H < 1$, and W_t is a Brownian motion process. By an application of the stochastic Fubini Theorem, one gets

$$\begin{aligned}\int_0^t \int_0^s (s-u+\varepsilon)^{\alpha-1}\mathrm{d}W_u \mathrm{d}s &= \int_0^t \int_u^t (s-u+\varepsilon)^{\alpha-1}\mathrm{d}s\mathrm{d}W_u \\ &= \frac{1}{\alpha}\int_0^t ((t-u+\varepsilon)^\alpha - \varepsilon^\alpha)\mathrm{d}W_u \\ &= \frac{1}{\alpha}\left[\int_0^t (t-u+\varepsilon)^\alpha \mathrm{d}W_u - \varepsilon^\alpha \int_0^t \mathrm{d}W_u\right] \\ &= \frac{1}{\alpha}[B_t^\varepsilon - \varepsilon^\alpha W_t].\end{aligned}$$

Consequently

$$B_t^\varepsilon = \alpha \int_0^t \varphi_s^\varepsilon \mathrm{d}s + \varepsilon^\alpha W_t$$