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A note on Itô formula for fractional Brownian sheet with Hurst parameters $H_1, H_2 \in (0, 1)$ [☆]

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Abstract

By using the white noise theory for fractional Brownian sheet, we give a new proof of the Itô formula for fractional Brownian sheet with arbitrary Hurst parameters $H_1, H_2 \in (0, 1)$. Our proof is based on the repeated application of the Itô formulas for one-parameter Gaussian process.

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1. Introduction

The fractional Brownian sheet $B^H = (B_z^H)$, $z \in \mathbb{R}^2$, with Hurst parameters $H = (H_1, H_2)$ is a centered Gaussian process with the covariance

$$\mathbb{E}(B_a^H B_b^H) = \prod_{i=1}^{2} \frac{1}{2}(|a_i|^{2H_i} + |b_i|^{2H_i} - |a_i - b_i|^{2H_i}),$$

where $a = (a_1, a_2)$ and $b = (b_1, b_2)$. The theory of stochastic calculus with respect to fractional Brownian sheet has recently been developed by several authors [see e.g. Kim (2006), Kim and Jeon (2006), Kim, Jeon, and Park (2008), Tudor and Viens (2003) and Tudor and Viens (2006)].

In this paper, we introduce four types of stochastic integrals by using the fractional white noise theory. As an application of these various stochastic surface integrals, we develop the basic elements of stochastic calculus related to fractional Brownian sheet. Among them, we prove an Itô formula for a fractional Brownian sheet by using iterated stochastic integrals. Several versions of the Itô formula for a fractional Brownian sheet can be found in the literature mentioned above.

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We give some notations that are used throughout the paper. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be two points in the rectangle $\mathbb{R}^2$.

- The notation $a \preceq b$ will denote the condition $a_1 \le b_1$ and $a_2 \le b_2$.
- The notation $a \trianglelefteq_1 b$ will denote the condition $a_1 \le b_1$ and $a_2 \ge b_2$.
- The notation $a \otimes b$ will denote the point (a_1, b_2).
- When $a \preceq b$, the notation $[a, b]$ will denote the rectangle $[a_1, b_1] \times [a_2, b_2]$,
- The notation $\mathbf{1}_{(a,b)}(x)$ will denote the rectangle $\prod_{i=1}^{2} \mathbf{1}_{(a_i,b_i)}(x_i)$, where the indicator function $\mathbf{1}_{(s_i,t_i)}(x_i)$ is given by

$$\mathbf{1}_{(s_i,t_i)}(x_i) = \begin{cases} 1 & \text{for } s_i \le x_i < t_i \\ -1 & \text{for } t_i \le x_i < s_i \\ 0 & \text{otherwise.} \end{cases}$$

- For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, let us set $x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i}$.
- The notation $\mathbb{R}_a^2$ will denote the rectangle $(-\infty, a]$.
- The notation $\mathbb{R}_z^2 \trianglelefteq_1 \mathbb{R}_z^2$ will denote the set $\{(a, b) \in \mathbb{R}_z^2 \times \mathbb{R}_z^2 : a \trianglelefteq_1 b\}$.

2. Preliminaries and Stochastic integrals

First we briefly mention some basic facts of the white noise theory for fractional Brownian sheet. Let $\mathcal{S}(\mathbb{R}^2)$ be the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^2$. We consider the white noise space $(\Omega, \mathbf{F}, \mathbb{P})$ as the underlying probability space, i.e., $\Omega := \mathcal{S}'(\mathbb{R}^2)$ is the space of tempered distributions and $\mathbb{P}$ is an unique probability measure such that for all $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\int_{S'(\mathbb{R}^2)} \mathrm{e}^{\mathrm{i}\langle \omega, f\rangle} \mathrm{d}\mathbb{P}(\omega) = \mathrm{e}^{-(1/2)\|f\|_{L^2(\mathbb{R}^2)}^2}.$$

We can extend $\langle \cdot, f\rangle$ to $f \in L^2(\mathbb{R}^2)$ and $\mathbb{E}[\langle \cdot, f\rangle\langle \cdot, g\rangle] = (f, g)_{L^2(\mathbb{R}^2)}$. For $f \in \mathcal{S}(\mathbb{R}^2)$, we define $I_{\pm}^{H_i} f : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, by

$$I_{\pm}^{H_i} f(x) = \begin{cases} \dfrac{1}{\Gamma(H_i - (1/2))} \displaystyle\int_0^\infty \frac{f(x \mp u\epsilon_i)}{u^{(3/2)-H_i}} \mathrm{d}u & \text{for } 1/2 < H_i < 1 \\ f & \text{for } H_i = \dfrac{1}{2} \\ \dfrac{(1/2) - H_i}{\Gamma(H_i + (1/2))} \displaystyle\int_0^\infty \frac{f(x) - f(x \mp u\epsilon_i)}{u^{(3/2)-H_i}} \mathrm{d}u & \text{for } 0 < H_i < 1/2, \end{cases} \tag{2.1}$$

where $\epsilon_1 = (1, 0)$ and $\epsilon_2 = (0, 1)$. Let $M_{\pm\pm}^{H} f(x) = I_{\pm}^{H_1}(I_{\pm}^{H_2}) f(x)$. Then a continuous version of $\langle \cdot, M_{--}^{H} \mathbf{1}_{(0,t)}\rangle$ is a fractional Brownian sheet with arbitrary Hurst parameters $H_1, H_2 \in (0, 1)$ on $(\Omega, \mathbf{F}, \mathbb{P})$.

Let $\mathbf{H}_n(x)$ and h_n be the nth Hermite polynomial and nth Hermite function, $n = 0, 1, \dots$, respectively. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ (with $\mathbb{N} = \{1, 2, \dots\}$), let us set $\mathbf{e}_\alpha(x_1, x_2) = \prod_{i=1}^{2} h_{\alpha_i}(x_i)$. Denote by $\mathbf{A}$ the set of all finite sequences $\mathbf{a} = (a_1, a_2, \dots, a_m)$ with $a_i \in \{0\} \cup \mathbb{N}$, $m = 1, 2, \dots$. For $\mathbf{a} \in \mathbf{A}$, we set $\mathbf{a}! = \prod_{i=1}^{\infty} a_i!$ and $|\mathbf{a}| = \sum_{i=1}^{\infty} a_i$. Let $\alpha^{(i)}$, $i = 1, 2, \dots$, be a fixed ordering of $\mathbb{N}^2$ such that for $i < j$, $|\alpha^{(i)}| \le |\alpha^{(j)}|$. With these notations, we define

$$\mathbf{H}_{\mathbf{a}}(\omega) = \prod_{i=1}^{\infty} \mathbf{H}_{a_i}(\langle \omega, \mathbf{e}_{\alpha^{(i)}}\rangle).$$

We recall the following chaos expansion theorem.

Theorem 2.1. *Let $F \in \mathbb{L}^2 := L^2(\Omega, \mathbf{F}, \mathbb{P})$. Then there exist constants $c_{\mathbf{a}} \in \mathbb{R}$ for $\mathbf{a} \in \mathbf{A}$ such that*

$$F(\omega) = \sum_{\mathbf{a} \in \mathbf{A}} c_{\mathbf{a}} \mathbf{H}_{\mathbf{a}}(\omega) \text{ limit in } \mathbb{L}^2. \tag{2.2}$$

Furthermore, we have the isometry $\|F\|_{\mathbb{L}^2}^2 = \sum_{\mathbf{a} \in \mathbf{A}} \mathbf{a}! c_{\mathbf{a}}^2$.