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Continuously dynamic additive models for functional data

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ABSTRACT

In this article, we propose the continuously dynamic additive model (CDAM), in which both the predictor and response are random functions. In continuously dynamic additive modeling, we assume that additivity occurs in the time domain rather than in spectral domain, and characterize this model through a time-dependent smooth surface that reflects the underlying nonlinear dynamic relationships between functional predictor and functional response. We use tensor product basis expansion with varying coefficient functions to approximate the time-varying smooth surface, and then estimate varying-coefficient functions by combining functional principal components analysis with penalized least squares method. In a theoretical investigation, we show that the predictions obtained from the fitted CDAM are asymptotically consistent under some mild conditions. Finally, we demonstrate the superiority of the proposed model and method through extensive simulation studies as well as a real data example.

1. Introduction

Additive models have been enjoying increased popularity over the last decade because they can provide effective dimension and great flexibility in modeling. There exists an extensive literature on additive models for many regression situations that involve continuous predictors and both continuous and generalized responses (e.g., Mammen and Pack [5], Wood [18], Yu et al. [22], Ravikumar et al. [14], Carroll et al. [2]).

Müller and Yao [10] proposed functional additive models for functional predictors and scalar response, where the functional principal components (FPC) of the predictor process were assumed to be additive. Utilizing the uncorrelatedness of the predictor FPC scores and a sequence of one-dimensional smoothing steps, they estimated the smooth component functions and studied asymptotic properties of the resulting models. Müller et al. [9] pursued a different kind of additivity. They assumed that additivity occurred in the time domain rather than in the spectral domain and developed continuously additive models for nonlinear functional regression. McLean et al. [6] described the functional generalized additive models for associates studies. By adopting tensor product B-splines with roughness penalties, they obtained estimates of the unknown regression functions and illustrated the usefulness of the approach through an application to brain tractography.

In practice, people are often interested in more general situations in which the predictor and the response are stochastic processes. For example, Müller et al. [7] introduced a functional volatility process for modeling volatility trajectories for high frequency observations and successfully predicted the trajectory of volatility for the second half day of trading by applying the ordinary functional linear model. For functional predictors and functional responses, the continuously additive models proposed by Müller et al. [9] cannot be used to deal with this kind of functional data. Although functional additive models

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introduced by Müller and Yao [10] are also suitable for functional response case, their proposed models are assumed to be independent in the predictor FPC scores.

In this article, we investigate additive models with functional predictors and functional responses, where we assume that additivity occurs in the time domain, rather that in the resulting FPC scores (Müller and Yao [10]), and then propose CDAM. We use tensor product basis expansion with varying coefficient functions to approximate the time-varying smooth additive surface that characterizes CDAM, and combine FPC with penalized least squares method to estimate the varying-coefficient functions in the approximating model. The fast and simple implementation makes it possible to study asymptotic properties of the resulting model. At last, in the theoretical investigation, we derive the asymptotic properties of estimation errors, and provide the rates of convergence of the estimated predictions obtained by applying CDAM.

For functional predictor and functional response, Ramsay and Dalzell [13] first introduced related works. Yao et al. [21] proposed nonparametric methods for sparse longitudinal data under the framework of functional linear regression. One more worth mentioning is that Scheipl et al. [16] developed functional additive mixed models for correlated functional responses, and provided the estimation formula based on orthonormal basis representation and standard additive mixed models. Although these models considered in Scheipl et al. [16] and Brockhaus et al. [1] include our proposed model as a special case, we try to tackle the dynamic additive model from the perspective of FPC analysis, and obtain the asymptotic properties of the proposed estimation methods and predicted trajectories.

Our proposed CDAM not only reflects the underlying dynamic relationships between predictors and response processes, but also is more flexible than the ordinary functional regression models. Furthermore, as we have demonstrated in simulation studies, CDAM performs better in the overall levels of error distributions than other comparison models when the underlying regression relationship is nonlinear. Meanwhile, the losses associated with our proposed CDAM are relatively small in linear cases.

The remainder of this article is organized as follows. In Section 2, CDAM is proposed, and the predicted trajectory is then obtained. Assumptions and asymptotic properties of the proposed estimators are described in Section 3. Simulation studies and an application of the proposed model to a real data analysis are conducted in Sections 4 and 5, respectively. Some discussions and concluding remarks are presented in Section 6. Technical details of the theoretical results are listed in the Supplementary Material 1 (see Appendix A). The estimation steps of CDAM used in the article are available in the Supplementary Material 2 (see Appendix A).

2. Continuously dynamic additive models

We consider a class of flexible functional nonlinear models in which the predictor X and the response Y are smooth, square-integrable random processes defined on a real bounded and closed interval pair $(\mathcal{J}_X, \mathcal{J}_Y)$, with unknown mean functions $\mu_X(s) = \mathrm{E}X(s)$, $\mu_Y(t) = \mathrm{E}Y(t)$. We propose the following continuously dynamic additive model (CDAM):

$$\mathrm{E}\{Y(t)|X\} = \mu_Y(t) + \int_{\mathcal{J}_X} g\{t, s, X(s)\}ds, \quad t \in \mathcal{J}_Y, \tag{2.1}$$

for a time-varying smooth additive surface $g(\cdot,\cdot,\cdot)$: $\mathcal{J}_Y \times \mathcal{J}_X \times \mathcal{R} \to \mathcal{R}$, which satisfies $\mathrm{E}[\int_{\mathcal{J}_X} g\{t, s, X(s)\}ds] = 0$ for all $t \in \mathcal{J}_Y$ for identifiability.

Similar to the continuously additive models for scalar responses in Müller et al. [9], the CDAM can be viewed as the limit of a sequence of time-varying additive models as the time grid $\{s_1, \ldots, s_{N_1}\} \in \mathcal{J}_X$ becomes increasingly dense. Taking the limit $N_1 \to \infty$ of the standardized additive models

$$\mathrm{E}\{Y(t)|X(s_1), \ldots, X(s_{N_1})\} = \mu_Y(t) + N_1^{-1}\sum_{j=1}^{N_1} g\{t, s_j, X(s_j)\}, \quad t \in \mathcal{J}_Y,$$

and replacing the sum by an integral, we yield the CDAM (2.1).

The classical functional linear model (FLM) defined by Yao et al. [21] arises from the choice $g\{t, s, X(s)\} = \{X(s) - \mu_X(s)\}\beta(s,t)$, where $\beta(s,t)$ is a smooth and square integrable regression function. When the response is a scalar, our proposed model (2.1) can be simplified to the continuously additive model for nonlinear functional regression introduced by Müller et al. [9]. Therefore, our model is flexible enough to cover a variety of situations and can be considered as an extension of additive models with vector predictors and scalar response to the case of functional predictors and functional response.

In the following, we assume that the entire predictor functions $\{X_i\}_{i=1}^n$ are fully observed. If predictor trajectories are densely observed with measurement errors, then one can implement a pre-processing smoothing step with local linear fitting to obtain continuous trajectories. Other smoothing techniques can be used equally well (see for instance Ramsay [12]).

From (2.1), it can be observed that CDAM is characterized by the time-varying smooth additive surface $g(t,\cdot,\cdot)$. For every $t \in \mathcal{J}_Y$ and any sets of orthonormal basis functions $\{\phi_l(\cdot)\}_{l=1}^\infty$ on the domain $\mathcal{J}_X$ and $\{\psi_k(\cdot)\}_{k=1}^\infty$ on the range of X, one can find varying coefficient functions $\gamma_{lk}(t)$ such that $g(t,\cdot,\cdot)$ in (2.1) can be represented as

$$g(t, s, x) = \sum_{l,k=1}^{\infty} \gamma_{lk}(t)\phi_l(s)\psi_k(x), \tag{2.2}$$