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Bivariate Conway–Maxwell–Poisson distribution: Formulation, properties, and inference

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ABSTRACT

The bivariate Poisson distribution is a popular distribution for modeling bivariate count data. Its basic assumptions and marginal equi-dispersion, however, may prove limiting in some contexts. To allow for data dispersion, we develop here a bivariate Conway–Maxwell–Poisson (COM–Poisson) distribution that includes the bivariate Poisson, bivariate Bernoulli, and bivariate geometric distributions all as special cases. As a result, the bivariate COM–Poisson distribution serves as a flexible alternative and unifying framework for modeling bivariate count data, especially in the presence of data dispersion.

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1. Introduction

Marshall and Olkin [15] used the bivariate Bernoulli distribution and exploited the well-known relationships between Poisson, Bernoulli, and geometric distributions (among others) to generate their respective bivariate analogs. In contrast, this work develops a bivariate Conway–Maxwell–Poisson (bivariate COM–Poisson) distribution as a flexible family of distributions that includes the bivariate Poisson, bivariate Bernoulli, and bivariate geometric distributions as special cases. With an added dispersion parameter, the bivariate COM–Poisson distribution proves to be a useful model for count data when data dispersion is present. The bivariate COM–Poisson distribution provides a unifying framework for defining common discrete bivariate distributions and, more generally, serves as a bridge distribution through the dispersion parameter to capture other structures displaying over- or under-dispersion.

This work derives a bivariate COM–Poisson distribution to serve as a flexible distribution for modeling bivariate count data in the presence of data dispersion (including its special-case bivariate distributions). Section 2 provides background

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regarding classical bivariate count distributions, and introduces the univariate form of the COM–Poisson distribution in order to motivate its bivariate analog. Section 3 defines the bivariate COM–Poisson distribution and outlines associated statistical properties of this distribution. Section 4 addresses matters of statistical inference, discussing issues regarding parameter estimation and hypothesis testing. Section 5 provides examples illustrating the flexibility of the bivariate COM–Poisson distribution for bivariate count datasets with differing levels and/or forms of dispersion present. Section 6 presents a further generalization to encompass additional bivariate distributions, namely the bivariate sum-of-COM–Poisson distribution. This generalization not only captures the bivariate COM–Poisson distribution (and hence all of its special cases), but also captures the bivariate negative binomial and bivariate binomial distributions as special cases. Finally, Section 7 concludes with some discussion.

2. Background

We highlight several well-known bivariate distributions discussed in [15], including the bivariate Poisson, bivariate Bernoulli, and bivariate geometric in Section 2.1. Meanwhile, Section 2.2 introduces the univariate COM–Poisson distribution which provides a basis for its bivariate generalization.

2.1. Some well-known bivariate distributions

Marshall and Olkin [15], Kocherlakota and Kocherlakota [11], and Johnson et al; [9] all discuss several bivariate forms of common distributions, including the bivariate Bernoulli, bivariate Poisson, and bivariate geometric distributions. The bivariate Bernoulli distribution is established in a simple manner by starting with the random pair (X, Y) having only four possible values—the possible combinations containing 0 and 1 [i.e., (0,0), (0,1), (1,0), and (1,1)]. Denoting the associated probabilities by p_{00}, p_{01}, p_{10}, and p_{11},

$$\begin{aligned}\Pi(t_1, t_2) &= 1 + p_{1+}(t_1 - 1) + p_{+1}(t_2 - 1) + p_{11}(t_1 - 1)(t_2 - 1)\\ &= p_{00} + p_{10}t_1 + p_{01}t_2 + p_{11}t_1t_2 \qquad (1)\end{aligned}$$

[11], where $p_{i+} = p_{i0} + p_{i1}$ and $p_{+i} = p_{0i} + p_{1i}$ (for $i = 0, 1$) are the associated marginal probabilities; this notation is used throughout the manuscript. Marshall and Olkin [15] refer to this pgf as the factorial moment generating function.

The bivariate Poisson distribution has been derived or obtained by numerous authors, including Maritz [13], M'Kendrick [17], and Teicher [22]; see [9,15] for comprehensive discussions in this regard. In particular, the trivariate reduction method is a natural and popular approach for the construction of bivariate discrete distributions, particularly the bivariate Poisson distribution. Here, one sets $X = X_1 + X_3$ and $Y = X_2 + X_3$, where the X_i's are independent $\mathcal{P}oi(\lambda_i)$ random variables, for $i = 1, 2, 3$; see Chapter 37 of Johnson et al. [9]. Accordingly, the joint pgf of (X, Y), as noted in [11,15], is

$$\Pi(t_1, t_2) = \exp\{(\lambda_1 + \lambda_3)(t_1 - 1) + (\lambda_2 + \lambda_3)(t_2 - 1) + \lambda_3(t_1 - 1)(t_2 - 1)\}. \qquad (2)$$

An alternative method for deriving the bivariate Poisson distribution is to compound the bivariate binomial distribution with a Poisson distribution. Let $(X, Y \mid n_*)$ denote a conditional bivariate binomial distribution (conditional on the number of trials, n_*). Accordingly, its joint pgf is

$$\Pi(t_1, t_2 \mid n_*) = \{1 + p_{1+}(t_1 - 1) + p_{+1}(t_2 - 1) + p_{11}(t_1 - 1)(t_2 - 1)\}^{n_*}, \qquad (3)$$

where $n_* \sim \mathcal{P}oi(\lambda_*)$. Thus, the unconditional joint pgf of (X, Y) is

$$\begin{aligned}\Pi(t_1, t_2) &= \sum_{n_*=0}^{\infty} \frac{\lambda_*^{n_*} e^{-\lambda_*}}{n_*!} \Pi(t_1, t_2 \mid n_*)\\ &= \exp\{\lambda_* p_{1+}(t_1 - 1) + \lambda_* p_{+1}(t_2 - 1) + \lambda_* p_{11}(t_1 - 1)(t_2 - 1)\}, \qquad (4)\end{aligned}$$

where, equating Eqs. (2) and (4) yields $\lambda_1 + \lambda_3 = \lambda_* p_{1+}$, $\lambda_2 + \lambda_3 = \lambda_* p_{+1}$, and $\lambda_3 = \lambda_* p_{11}$ [11].

Finally, Marshall and Olkin [15] define the bivariate geometric distribution as one having joint pgf

$$\Pi(t_1, t_2) = \mathrm{E}(t_1^X t_2^Y) = \frac{p_{11} - (t_1 + t_2)\tau - t_1t_2(p_{01}p_{10} - p_{00}\tau)}{(1 - p_{0+}t_1)(1 - p_{+0}t_2)(1 - p_{00}t_1t_2)},$$

where τ and ρ are defined as

$$\mathrm{Cov}(X, Y) = p_{11} - p_{1+}p_{+1} \doteq \tau, \quad \text{and} \quad \mathrm{Corr}(X, Y) = \frac{p_{11} - p_{1+}p_{+1}}{\sqrt{p_{1+}p_{0+}p_{+1}p_{+0}}} \doteq \rho.$$