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Conditioned limit laws for inverted max-stable processes

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ABSTRACT

Max-stable processes are widely used to model spatial extremes. These processes exhibit asymptotic dependence meaning that the large values of the process can occur simultaneously over space. Recently, inverted max-stable processes have been proposed as an important new class for spatial extremes which are in the domain of attraction of a spatially independent max-stable process but instead they cover the broad class of asymptotic independence. To study the extreme values of such processes we use the conditioned approach to multivariate extremes that characterises the limiting distribution of appropriately normalised random vectors given that at least one of their components is large. The current statistical methods for the conditioned approach are based on a canonical parametric family of location and scale norming functions. We study broad classes of inverted max-stable processes containing processes linked to the widely studied max-stable models of Brown–Resnick and extremal-t, and identify conditions for the normalisations to either belong to the canonical family or not. Despite such differences at an asymptotic level, we show that at practical levels, the canonical model can approximate well the true conditional distributions.

1. Introduction

Extreme environmental events, such as hurricanes, heatwaves, flooding and droughts, can cause havoc to the people affected and typically result in large financial losses. The impact of this type of event is often exacerbated by the event being severe over a large spatial region. The statistical modelling of spatial extremes is a rapidly evolving area [6] and is crucial to understanding, visualising and predicting the extremes of stochastic processes. The approach that is currently most used for modelling spatial extreme values assumes the environmental process is a max-stable process [8]. The most widely used max-stable processes are the Brown–Resnick [2,18,15], and extremal-t [26,27] processes. These include the Smith process [34] and extremal Gaussian process [33] as special cases.

Max-stable processes are the only non-trivial limits of point-wise normalised maxima of independent and identically distributed realisations of stochastic processes. When max-stable processes are observed at a finite number of locations their joint distribution is a multivariate extreme value distribution, which is underpinned by the assumption of the original variables satisfying the dependence structure conditions of multivariate regular variation [30]. Max-stable processes have marginal generalised extreme value distributions [3] and a complex non-negative dependence structure which has a restricted form. To understand this restriction, let $\{X_M(s),\ s \in \mathbb{R}^2\}$ be a spatial max-stable process with continuous marginal

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distribution functions F_s and corresponding inverse denoted by $F_s^{\leftarrow}$. Then, if for any $s_1, s_2 \in \mathbb{R}^2$ where $X_M(s_1)$ and $X_M(s_2)$ are not independent, it follows that the dependence coefficient

$$\chi = \lim_{p\to 1} \Pr\left\{X_M(s_2) > F_{s_2}^{\leftarrow}(p) \mid X_M(s_1) > F_{s_1}^{\leftarrow}(p)\right\},$$

is positive. This property is termed asymptotic dependence, and it implies that the largest marginal values at different locations can occur simultaneously. Furthermore, if we denote the joint distribution function of $\{X_M(s_1), X_M(s_2)\}$ by F_{s_1,s_2} then

$$F_{s_1,s_2}\{F_{s_1}^{\leftarrow}(p), F_{s_2}^{\leftarrow}(p)\} = p^{\theta(s_1,s_2)}$$

for all $0 < p \le 1$ where $1 \le \theta(s_1, s_2) \le 2$, so for max-stable processes this implies that the spatial dependence properties, measured by θ are independent of the severity of the event, measured by p [6].

Wadsworth and Tawn [38] introduced the class of inverted max-stable processes and these are used in Davison et al. [5] and Thibaud et al. [37]. Any inverted max-stable process $\{X(s),\ s \in \mathbb{R}^2\}$ with unit exponential margins, i.e., for all $s \in \mathbb{R}^2$ and $x > 0$, $\Pr(X(s) < x) = 1 - \exp(-x)$, can be represented by

$$X(s) = 1/X_F(s) \quad s \in \mathbb{R}^2,$$

where $\{X_F(s), s \in \mathbb{R}^2\}$ is a max-stable process with unit Fréchet margins, i.e., for all $s \in \mathbb{R}^2$ and $x > 0$, $\Pr(X_F(s) < x) = \exp(-1/x)$. Thus, for all $s_1, s_2 \in \mathbb{R}^2$, the dependence structure between large $X(s_1)$ and $X(s_2)$ is equivalent to the dependence structure between small $X_F(s_1)$ and $X_F(s_2)$ and hence differs from the max-stable form. As all max-stable processes are non-negatively associated, so are all inverted max-stable processes. Furthermore, for all $s_1, s_2 \in \mathbb{R}^2$, with $s_1 \ne s_2$, for $x > 0$

$$\Pr\{X(s_1) > x, X(s_2) > x\} = \exp\{-x/\eta(s_1, s_2)\} = \{\Pr(X > x)\}^{1/\eta(s_1,s_2)},$$

where $\eta := \eta(s_1, s_2) \in [1/2, 1)$ is the coefficient of tail dependence [23]. It follows that in general margins the inverted max-stable distribution is

$$F_{s_1,s_2}\{F_{s_1}^{\leftarrow}(p), F_{s_2}^{\leftarrow}(p)\} = 1 - 2(1-p) + (1-p)^{1/\eta} \sim p^{2-(1-p)^{-1+1/\eta}} \quad \text{as } p \uparrow 1,$$

so unlike for the max-stable distribution the spatial dependence, measured by $2 - (1-p)^{-1+1/\eta}$, is not independent of the severity p. In fact, all non-perfectly dependent inverted max-stable processes are in the domain of attraction of spatially independent max-stable processes, meaning that their point-wise normalised maxima are independent, i.e., for all $s_1, s_2 \in \mathbb{R}^2$, with $s_1 \ne s_2$,

$$\lim_{n\to\infty} \Pr\left(\max_{i=1,\dots,n} X_i(s_1) - \ln n < x_1,\ \max_{i=1,\dots,n} X_i(s_2) - \ln n < x_2\right) = \prod_{i=1}^{2} \exp\{-\exp(-x_i)\}, \tag{1}$$

for any $x_1, x_2 \in \mathbb{R}$, where $\{X_i(s), s \in \mathbb{R}^2\}$, for $i = 1, \dots, n$ denotes a sequence of independent and identically distributed inverted max-stable processes with unit exponential margins.

To reveal the extremal dependence structure for asymptotically independent random variables, alternative asymptotic properties have been studied. Ledford and Tawn [22,23], Resnick [31] and Wadsworth and Tawn [39] explore how the joint survivor decays as both arguments tend to the upper end point. A weakness with this approach is that it fails to describe the behaviour of the $X(s_2)$ values that occur with the largest values of $X(s_1)$. Instead a conditioned approach is required which looks at a more subtle normalisation for $X(s_2)$ that focuses on the region of the joint distribution which is most likely when conditioning on variable $X(s_1)$ being large. This is the approach we take in this paper.

For a bivariate random variable (X, Y) with unit exponential margins and general dependence structure, the conditioned extremes limit theory of Heffernan and Tawn [13] and Heffernan and Resnick [12] is equivalent to the assumption that there exist location and scaling norming functions $a : \mathbb{R}_+ \mapsto \mathbb{R}$ and $b : \mathbb{R}_+ \mapsto \mathbb{R}_+$, such that, for any $x > 0$ and $z \in \mathbb{R}$,

$$\lim_{u\to\infty} \Pr\{X - u > x, \{Y - a(X)\}/b(X) < z \mid X > u\} = \exp(-x)G(z), \tag{2}$$

where G is a non-degenerate distribution function. To ensure a, b and G are uniquely defined the condition $\lim_{z\to\infty} G(z) = 1$ is required, so G places no mass at $+\infty$ but some mass is allowed at $-\infty$.

The key development in this paper is deriving the forms of a, b and G in representation (2) for the broad class of inverted max-stable processes. For positively dependent random variables, Heffernan and Tawn [13] found that, for all the standard copula models studied by Joe [17] and Nelsen [25], the norming functions $a(x)$ and $b(x)$, fell into the simple canonical parametric family

$$a(x) = \alpha x \quad \text{and} \quad b(x) = x^{\beta}, \tag{3}$$

where $\alpha \in [0, 1]$ and $\beta \in (-\infty, 1)$ with Keef et al. [19] identifying additional joint constraints. The case $\alpha = 1$ and $\beta = 0$ corresponds to $\chi > 0$, whereas any other combination of α and β gives $\chi = 0$. The statistical use of the canonical family (3)