Journal of Multivariate Analysis 145 (2016) 37–43

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Multivariate stochastic comparisons of multivariate mixture models and their applications

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A R T I C L E I N F O

Article history:
Received 2 May 2013
Available online 4 December 2015

AMS subject classifications:
62E10
62H05
62P30

Keywords:
Mixture models
Multivariate hazard rate ordering
Multivariate likelihood ratio ordering
$(n-k+1)$-out-of-n systems
Residual life-lengths of live components

A B S T R A C T

In this paper, we obtain some conditions to compare multivariate mixture models with respect to some well-known multivariate stochastic orders. We also utilize the established results in reliability theory to compare the vectors of residual life-lengths of live components of $(n-k+1)$-out-of-n systems in both one sample and two samples situations.

1. Introduction

Let Θ_1 (mixing random variable) be a random variable with distribution function Λ_1, centered on $\mathcal{X} \subseteq \mathbb{R}$. Suppose also that $U_i(\theta)$ (mixed random variables) are independent random variables such that $U_i(\theta)$ have survival and density functions $\bar{F}_i(.|\theta)$ and $f_i(.|\theta)$, respectively, for all $i = 1, \ldots, r$ and $\theta \in \mathcal{X}$. Then, the joint survival function of $\boldsymbol{U}(\Theta_1) = (U_1(\Theta_1), \ldots, U_r(\Theta_1))$ is given by

$$\bar{F}(x_1, \ldots, x_r) = \int_{\mathcal{X}} \prod_{i=1}^{r} \bar{F}_i(x_i|\theta) d\Lambda_1(\theta), \quad (x_1, \ldots, x_r) \in \mathbb{R}^r. \tag{1.1}$$

A model with survival function of the form in Eq. (1.1) is known as a multivariate mixture model. Mixture models appear naturally in many areas of applied probability and statistics such as reliability, survival analysis, and risk theory. Two examples in reliability theory are as follows (for more examples, interested readers may refer to Ma [13], Gerardi et al. [8], Belzunce et al. [6], and Misra and Misra [16], and the references therein):

- Let $T_1, \ldots, T_n$ be i.i.d. random variables, having a common absolutely continuous distribution, corresponding to the lifetimes of n components of a coherent system (see [5] for the definition of a coherent system). Samaniego [17] showed

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http://dx.doi.org/10.1016/j.jmva.2015.11.013

that the survival function of the lifetime of such a system can be expressed as a finite mixture of survival functions of k-out-of-n systems (a system which works if and only if at least k of its components work). Specifically, if T is the lifetime of the coherent system, then

$$P(T > t) = \sum_{k=1}^{n} p_k P(T_{k:n} > t),$$

where $T_{k:n}$ is kth order statistic from the random sample $T_1, \ldots, T_n$, and the probabilities $p_k,\ k = 1, \ldots, n$, are the elements of the signature vector of the system;
- Consider an $(n-k+1)$-out-of-n system comprising components with i.i.d. lifetimes. Then, Bairamov and Arnold [3] showed that the joint survival function of the residual life-lengths of the live components after the kth failure in the system can be expressed as a multivariate mixture model.

Multivariate stochastic comparisons of multivariate mixture models have been investigated extensively in the literature. We refer the readers to Misra et al. [15], Belzunce et al. [6], Khaledi and Shaked [10], Li and Da [12], Misra and Misra [16], Badía et al. [2], and Amini-Seresht and Khaledi [1] for review of the established results in this direction. Consider the random vector $\boldsymbol{V}(\Theta_2) = (V_1(\Theta_2), \ldots, V_r(\Theta_2))$ following the mixture model with joint survival function

$$\bar{G}(x_1, \ldots, x_r) = \int_{\mathcal{X}} \prod_{i=1}^{r} \bar{G}_i(x_i|\theta) d\Lambda_2(\theta), \quad (x_1, \ldots, x_r) \in \mathbb{R}^r, \tag{1.2}$$

wherein the mixed random variables $V_i(\theta)$ have survival and density functions $\bar{G}_i(.|\theta)$ and $g_i(.|\theta)$, respectively, for all $i = 1, \ldots, r$ and $\theta \in \mathcal{X}$, and the mixing random variable Θ_2 has distribution function Λ_2, centered on $\mathcal{X}$. In the present paper, we introduce some conditions which enable us to compare the random vector $\boldsymbol{U}(\Theta_1)$, described above, to the random vector $\boldsymbol{V}(\Theta_2)$ with respect to multivariate likelihood ratio and multivariate hazard rate orders. Furthermore, for the case when $\bar{G}_i(.|\theta)$ do not depend on θ for all $i = 1, \ldots, r$, it is shown that multivariate likelihood ratio, multivariate hazard rate and usual multivariate stochastic orders between $\boldsymbol{U}(\Theta_1)$ and $\boldsymbol{V}(\Theta_2)$ can be obtained by assuming the corresponding univariate orders between $U_i(\theta)$ and $V_i(\theta)$. As an application, we apply the obtained results in reliability theory to compare the vectors of residual life-lengths of live components of $(n-k+1)$-out-of-n systems in both one sample and two samples situations.

The rest of this paper is organized as follows. In Section 2, the definitions and notation pertinent to some aging properties and stochastic orderings are stated. Ordering results of two random vectors following the multivariate mixture models are established in Section 3. Finally, in Section 4, we present an application of the results given in Section 3.

2. Definitions and notation

In this section, we recall the definitions and notation pertinent to some aging properties and stochastic orderings that are used throughout the paper. As usual, the terms *increasing* and *decreasing* are used for *non-decreasing* and *non-increasing*, respectively. For any two real numbers x and y, we denote $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. Further, for any two vectors $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we adopt the following notation: $\boldsymbol{x} \vee \boldsymbol{y} = (x_1 \vee y_1, \ldots, x_n \vee y_n)$, $\boldsymbol{x} \wedge \boldsymbol{y} = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$, $\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \ldots, x_n + y_n)$, and $\boldsymbol{x} \geq (\leq) \boldsymbol{y}$ if and only if $x_i \geq (\leq) y_i$ for $i = 1, \ldots, n$.

First, we introduce some well-known concepts of univariate aging properties.

Definition 2.1. Consider a non-negative random variable X with absolutely continuous distribution function F, survival function $\bar{F} = 1 - F$, and density function f. Then:

(i) We say that X is new worse than used (NWU) if

$$\bar{F}(x)\bar{F}(t) \leq \bar{F}(x+t) \quad \text{for all } x, t \geq 0. \tag{2.3}$$

If the inequality in Eq. (2.3) is reversed, then X is said to be new better than used (NBU);

(ii) We say that X has an increasing hazard rate (IHR) if $\bar{F}(x+t)/\bar{F}(x)$ is a decreasing function of x for all $t \geq 0$. Similarly, X has a decreasing hazard rate (DHR) if $\bar{F}(x+t)/\bar{F}(x)$ is an increasing function of x for all $t \geq 0$;

(iii) We say that X has an increasing likelihood ratio (ILR) if $f(x+t)/f(x)$ is a decreasing function of x for all $t \geq 0$. Similarly, X has a decreasing likelihood ratio (DLR) if $f(x+t)/f(x)$ is an increasing function of x for all $t \geq 0$.

The following implications between all these aging properties are well-known:

$$ILR\ (DLR) \Longrightarrow IHR\ (DHR) \Longrightarrow NBU\ (NWU).$$

In the literature, the ILR and DLR properties are also known as the log-concave and log-convex properties, respectively. For comprehensive discussion on these and other univariate aging properties and relations between them, one may refer to Lai and Xie [11], and Marshall and Olkin [14].

Now, we briefly describe some notions of multivariate stochastic orderings that are most pertinent to the results established in the subsequent sections.