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Direct shrinkage estimation of large dimensional precision matrix

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In this work we construct an optimal shrinkage estimator for the precision matrix in high dimensions. We consider the general asymptotics when the number of variables $p \to \infty$ and the sample size $n \to \infty$ so that $p/n \to c \in (0, +\infty)$. The precision matrix is estimated directly, without inverting the corresponding estimator for the covariance matrix. The recent results from random matrix theory allow us to find the asymptotic deterministic equivalents of the optimal shrinkage intensities and estimate them consistently. The resulting distribution-free estimator has almost surely the minimum Frobenius loss. Additionally, we prove that the Frobenius norms of the inverse and of the pseudo-inverse sample covariance matrices tend almost surely to deterministic quantities and estimate them consistently. Using this result, we construct a bona fide optimal linear shrinkage estimator for the precision matrix in case $c < 1$. At the end, a simulation is provided where the suggested estimator is compared with the estimators proposed in the literature. The optimal shrinkage estimator shows significant improvement even for non-normally distributed data.

1. Introduction

The estimation of the covariance matrix, as well as its inverse (the precision matrix), plays an important role in many disciplines from finance and genetics to wireless communications and engineering. In fact, having a suitable estimator for the precision matrix we are able to construct a good estimator for different types of optimal portfolios (see [44,19]). Similarly, in the array processing, the beamformer or the so-called minimum variance distortionless response spatial filter is defined in terms of the precision matrix (see e.g., [58]). In practice, however, the true precision matrix is unknown and a feasible estimator, constructed from data, must be used.

If the number of variables p is much smaller than the sample size n we can use the sample estimator which is biased but a consistent estimator for the precision matrix (see e.g., [7]). This case is known in the multivariate statistics as the "standard asymptotics" (see [41]). There are many findings on the estimation of the precision matrix when a particular distribution assumption is imposed. For example, the estimation of the precision matrix under the multivariate normal distribution was considered by Krishnamoorthy and Gupta [38], Gupta and Ofori-Nyarko [32–34], Kubokawa [39] and Tsukuma and Konno [57]. The results in the case of multivariate Pearson type II distribution as well as the multivariate

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elliptically contoured stable distribution are obtained by Sarr and Gupta [50] as well as by Bodnar and Gupta [9] and Gupta et al. [35], respectively.

Unfortunately, in practice p is often comparable in size to n or even is greater than n, i.e., we are in the situation when both the sample size n and the dimension p tend to infinity but their ratio keeps (tends to) a positive constant. This case often arises in finance when the number of assets is comparable or even greater than the number of observations for each asset. Similarly, in genetics, the data set can be huge comparable to the number of patients. Both examples illustrate the importance of the results obtained for $p, n \to \infty$.

We deal with this type of asymptotics, called the "large dimensional asymptotics" and also known as the "Kolmogorov asymptotics", in the present paper. More precisely, it is assumed that the dimension $p \equiv p(n)$ is a function of the sample size n and $p/n \to c \in (0, +\infty)$ as $n \to \infty$. This general type of asymptotics was intensively studied by several authors (see [26,27,12] etc.). In this asymptotics the usual estimators for the precision matrix perform poorly and are not consistent anymore. There are some techniques which can be used to handle the problem. Assuming that the covariance (precision) matrix has a sparse structure, significant improvements have already been achieved (see [13,14,16]). For the low-rank covariance matrices see the work of Rohde and Tsybakov [47]. An interesting nonparanormal graphic model was recently proposed by Xue and Zou [60]. Also, in order to estimate the large dimensional covariance matrix the method of block thresholding can be applied (see [15]). If the covariance matrix has a factor structure then the progress has been made by Fan et al. [20].

However, if neither the assumption about the structure of covariance (precision) matrix nor about a particular distribution is imposed, not many results are known in the literature which are based on the shrinkage estimators in high-dimensional setting (cf. [42,40,10,59]). The shrinkage estimator was first developed by Stein [55] and forms a linear combination of the sample estimator and some target. The corresponding shrinkage coefficients are often called shrinkage intensities. Ledoit and Wolf [42] proposed to shrink the sample covariance matrix to the identity matrix and showed that the resulting estimator is well-behaved in large dimensions. This estimator is called the linear shrinkage estimator because it shrinks the eigenvalues of the sample covariance matrix linearly. Recently, Bodnar et al. [10] proposed a generalization of the linear shrinkage estimator, where the shrinkage target was chosen to be an arbitrary nonrandom matrix and they showed the almost sure convergence of the derived estimator to its oracle.

The aim of our paper is to construct a feasible estimator for the precision matrix using the linear shrinkage technique and random matrix theory. In contrast to well-known procedures, we shrink the inverse of the sample covariance matrix itself instead of shrinking the sample covariance matrix and then inverting it. The direct shrinkage estimation of the precision matrix can be used in several important practical situations where the application of the inverse of the shrinkage estimator of the covariance matrix does not perform well. For instance, this could happen when the data generating process follows a factor model which is very popular in economics and finance (cf. [4,20,22,21]). In this case the largest eigenvalue of the covariance matrix is of order p and, consequently, the inverse of the linear shrinkage estimator for the covariance matrix does not work well. In the case when $c > 1$ the pseudo inverse of the sample covariance matrix is taken. The recent results from random matrix theory allow us to find the asymptotics of the optimal shrinkage intensities and estimate them consistently.

Random matrix theory is a very fast growing branch of probability theory with many applications in statistics. It studies the asymptotic behavior of the eigenvalues of the different random matrices under general asymptotics (see e.g., [1,8]). The asymptotic behavior of the functionals of the sample covariance matrices was studied by Marčenko and Pastur [43], Yin [61], Girko and Gupta [28–30], Silverstein [51], Bai et al. [5], Bai and Silverstein [8], Rubio and Mestre [48] etc.

We extend these results in the present paper by establishing the almost sure convergence of the optimal shrinkage intensities and the Frobenius norm of the inverse sample covariance matrix. Moreover, we construct a general linear shrinkage estimator for the precision matrix which has *almost surely* the smallest Frobenius loss when both the dimension p and the sample size n increase together and $p/n \to c \in (0, +\infty)$ as $n \to \infty$. Additionally, we provide a bona fide optimal linear shrinkage estimator for the precision matrix in case $c < 1$.

The suggested approach can potentially be applied in functional data analysis (cf. [46,24,37,11,17]). For instance, Ferraty et al. [23] pointed out that functional data can be seen as a special case of a high-dimensional vector. This point has been further explored by Aneiros and Vieu [2,3]. The estimation of the covariance (precision) matrix of this high-dimensional vector can be used in determining the prediction for the dependent variable as well as the corresponding predictive design points.

The rest of the paper is organized as follows. In Section 2 we present some preliminary results from random matrix theory and formulate the assumptions used throughout the paper. In Section 3 we construct the *oracle* linear shrinkage estimator for the precision matrix and verify the main asymptotic results about the shrinkage intensities and the Frobenius norm of the inverse and pseudo-inverse sample covariance matrices. Section 4 is dedicated to the *bona fide* linear shrinkage estimator for the precision matrix while Section 5 contains the results of the simulation study. Here, the performance of the derived estimator is compared with other known estimators for the large dimensional precision matrices. Section 6 includes the summary, while the proofs of the theorems are presented in the supplementary material (Section 7).

2. Assumptions and notations

The "large dimensional asymptotics" or "Kolmogorov asymptotics" include $\frac{p}{n} \to c \in (0, +\infty)$ as both the number of variables $p \equiv p(n)$ and the sample size n tend to infinity. In this case the traditional sample estimator performs poorly or