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Gap between orthogonal projectors—Application to stationary processes

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In the statistics of processes, the usual convergence of projectors does not provide a good idea of the continuity of some phenomena. In order to address it, we introduce a measure of the gap between two projectors, that we link to another type of convergence of sequences of projectors. This allows to define the gap between two spectral measures and then to develop various properties associated with the latter. In particular, we establish the continuity of the convolution product. A study of the closeness between two continuous random functions and the exhibition of a common filter for them illustrate the possible applications in functional statistics.

1. Introduction

One of the aims of this text is to introduce a notion of proximity between projectors on a $\mathbb{C}$-Hilbert space H and hence to establish a new concept of proximity between spaces generated by stationary processes.

As H can be an infinite dimensional space of random functions, this study may be applied to compare sets of curves, each curve being the observation of a random stationary process. We specify this idea below.

Let G be a locally compact abelian group, as, for example, $\mathbb{Z}^k$ or $\mathbb{R}^k$, where k is a positive integer. In the following, we define a stationary continuous random function (c.r.f.) as a random function from G into a Hilbert space H, which we denote by $X = (X_g)_{g\in G}$. The stationarity is considered in its weak definition. The randomness of this function comes from the fact that H can be a probability space as $L^2(\Omega)$, or $L^2_{\mathbb{C}^p}(\Omega)$. A set of observations of $X = (X_g)_{g\in G}$ for a subset $\{w_j,\ j \in J\}$ of Ω is then a set of functional data $\{X^j;\ j \in J\}$. When the subset $\{w_j,\ j \in J\}$ indexes consecutive time intervals, such as hours or days, and when $G = \mathbb{Z}$, we speak about functional time series. When $G = \mathbb{Z}^k$, we get observations of spatial processes. Horváth and Kokoszka [14] provide a large place for the analysis of these two particular kinds of functional data, and give a summary of recent advances. The main works on such data consist in forecasting functional time series. Among the first advances in this domain, we can find Masry [19], and Aneiros Pérez and Vieu [1], for mixed processes. Hyndman and Yasmeen [16]

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use of Principal Components Analysis for forecasting. Other works concern the development of tests for comparison of time series (see [15]). More general forms of dependence have been considered by Benhenni, Hedli-Griche, Rachdi and Vieu [2], and more recently by Ling, Liang and Vieu [18]. These inferential developments, following a large literature on functional data, as the monographs of Ramsay and Silverman [21], Ferraty and Vieu [12], Bosq [4] and Bongiorno, Goia, Salinelli and Vieu [3], consider time series in the time domain. For this, the time dependence effects have to be considered. A discussion on the importance of this dependence can be founded in [14]. It is clear that, in the time domain, the stationarity lets take into account any kind of dependence between two any times.

The originality of this work is to get rid of this problem of time-dependence for functional time series or more general objects, working on frequency-based objects. Indeed, if $X = (X_g)_{g\in G}$ is a stationary c.r.f., there exists a group of unitary operators $\{U_g;\ g \in G\}$ and an element X_0 of H, such that, for each g of G, $X_g = U_gX_0$ [5]. This group is linked with the spectral measure associated with the stationary c.r.f., and the object of this paper is to define a gap between two spectral measures which can retrieve the proximity between two stationary c.r.f.'s. The problem of estimation is still open, as the applications were widely open for Dauxois, Pousse and Romain [10], where can be founded some of the bases of the functional Principal Components Analysis.

For a comparison in concrete cases, the step which remains to be developed is the estimation of the associated unitary operators $\{U_g;\ g \in G\}$, and of an initial function X_0 such that we can write the model

$$X_g = U_gX_0 + \varepsilon_g, \quad g \in G,$$

where ε_g is a residual random variable, and for which we have the observations

$$X_g(w_j) = \widehat{U_g}X_0(w_j) + \varepsilon_g(w_j), \quad j \in J,\ g \in G.$$

Then, for example, considering two sets of curves $\{X^j;\ j \in J\}$ and $\{X'^j;\ j \in J\}$, we could estimate the sets of unitary operators $\{U_g;\ g \in G\}$ and $\{U'_g;\ g \in G\}$, the initial functions X_0 and X'_0, and get an estimation of the common part for the two sets of operators, and then of the two sets of curves.

In Section 2, we define a new type of convergence of projectors. This convergence is based on a relation of partial order defined on the set of orthogonal projectors on H.

We then study the gap $d(P, P')$ between two projectors P and P'. It is defined thanks to the above partial relation of order. We recall that the study of projectors takes an important part in Statistics and Probability (cf. [23]).

In Section 3, we study the projector-valued spectral measures. These measures are defined as applications from a σ-field ξ, which is classical for such mathematical objects as measures, into the set of projectors. For this purpose, we use the results obtained in Section 2. Then, we show that for any spectral measure $\mathcal{E}$, if $(A_n)_{n\in\mathbb{N}}$ is a sequence of elements of the σ-field ξ which converges to A, the sequence of projectors $(\mathcal{E}A_n)_{n\in\mathbb{N}}$ converges to $\mathcal{E}A$, result which is not true in $\mathcal{L}(H)$. Let us recall that the spectral measure, sometimes named decomposition of the unity, is a tool very often used in analysis. It allows the representation of positive operators and unitary operators by means of integrals (cf. [9,11,22,7]).

Considering two spectral measures $\mathcal{E}_1$ and $\mathcal{E}_2$, it seems natural to define the gap between these measures by the upper bound of the family of projectors $\{d(\mathcal{E}_1A, \mathcal{E}_2A), A \in \xi\}$. Then, we show that the convolution product of two spectral measures, as defined in [7], is continue relatively to this measure of the gap between two spectral measures.

In Section 4, we define and study the maximal equalizator of two unitary operators. We can roughly define it as the projector on the greater vector subspace on which the two unitary operators are equal. We can show that this projector is the orthogonal of the projector which measures the gap between the two spectral measures, respectively associated with the two unitary operators.

All through this paper, H will designate a $\mathbb{C}$-Hilbert space, and $\mathcal{P}(H)$ the family of the orthogonal projectors. Any projector P is supposed to be orthogonal, and we denote by $P^\perp$ the projector $I - P$. Several properties are obtained through transformation to orthogonal. When H is a space of random variables of type $L^2(\Omega, \mathcal{A}, P)$, the tools developed are useful for the statistics of stationary processes. Let us recall that $\|A\|_{\mathcal{L}} = \sup\{\|Ax\|; \|x\| = 1\}$ is a norm which equips the space $\mathcal{L}(H)$ of the bounded endomorphisms of H with a structure of Banach space.

2. Relation of partial order and proximity between projectors

The aim of this section is to study a well-known relation of partial order, defined on the set $\mathcal{P}(H)$ and noted $\ll$. For this relation, any set $\{P_\lambda;\ \lambda \in \Lambda\}$ of projectors, finite or infinite, countable or not, has got a greater minorant and a smaller majorant, and so a lower bound, $\inf\{P_\lambda;\ \lambda \in \Lambda\}$, and an upper bound, $\sup\{P_\lambda;\ \lambda \in \Lambda\}$. It is then possible to define the limits inferior and superior of a sequence of projectors $(P_n)_{n\in\mathbb{N}}$: $\liminf(P_n)_{n\in\mathbb{N}} = \sup\{\inf\{P_m; m \geq n\};\ n \in \mathbb{N}\}$ and $\limsup(P_n)_{n\in\mathbb{N}} = \inf\{\sup\{P_m; m \geq n\};\ n \in \mathbb{N}\}$. Of course we can prove that $\liminf(P_n)_{n\in\mathbb{N}} \ll \limsup(P_n)_{n\in\mathbb{N}}$, and when there is equality, we say that the sequence r-converges to $\liminf(P_n)_{n\in\mathbb{N}} = \limsup(P_n)_{n\in\mathbb{N}}$.

Then we study a gap between two projectors P and P' defined by $d(P, P') = \sup\{P, P'\} - \inf\{P, P'\}$. Of course, although this pseudo-distance is a projector, it is very similar, for its definition and its properties, with a distance. In particular, we show that a sequence of projectors $(P_n)_{n\in\mathbb{N}}$ r-converges to a projector P if and only if the sequence of projectors $(d(P_n, P))_{n\in\mathbb{N}}$ r-converges to 0.

We establish dual properties such that $P \ll Q$ if and only if $Q^\perp \ll P^\perp$, $(\sup\{P_\lambda;\ \lambda \in \Lambda\})^\perp = \inf\{P_\lambda^\perp;\ \lambda \in \Lambda\}$ or even more $(\limsup\{P_\lambda;\ \lambda \in \Lambda\})^\perp = \liminf\{P_\lambda^\perp;\ \lambda \in \Lambda\}$, which play an important role in the proofs.