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Asymptotics for characteristic polynomials of Wishart type products of complex Gaussian and truncated unitary random matrices

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ABSTRACT

Based on the multivariate saddle point method we study the asymptotic behavior of the characteristic polynomials associated to Wishart type random matrices that are formed as products consisting of independent standard complex Gaussian and a truncated Haar distributed unitary random matrix. These polynomials form a general class of hypergeometric functions of type ${}_2F_r$. We describe the oscillatory behavior on the asymptotic interval of zeros by means of formulae of Plancherel–Rotach type and subsequently use it to obtain the limiting distribution of the suitably rescaled zeros. Moreover, we show that the asymptotic zero distribution lies in the class of Raney distributions and by introducing appropriate coordinates elementary and explicit characterizations are derived for the densities as well as for the distribution functions.

1. Introduction

The behavior of the eigenvalues of random matrices is a large subject of research in random matrix theory. Recently, the study of products of random matrices gained particular interest (see, e.g., [1,2,4,6,7,11–13,17]). Let $r \in \mathbb{N}$ be an arbitrary positive integer and denote by $G_1, \ldots, G_r$ independent standard complex Gaussian random matrices (matrices of this kind are called Ginibre random matrices). Moreover, let each matrix G_j be of dimension $N_j \times N_{j-1}$ and let the matrix Y_r be defined as the product

$$Y_r = G_r \cdots G_1. \tag{1.1}$$

Assuming $N_0 = \min\{N_0, \ldots, N_r\}$ and writing $n = N_0$, let us consider the $n \times n$-dimensional matrix $Y_r^* Y_r$, where Y_r^* denotes the conjugate transpose of Y_r. It was shown by Akemann et al. [3] that the eigenvalues of $Y_r^* Y_r$ form a determinantal point

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process with a correlation kernel expressible in terms of Meijer G-functions. Moreover, Kuijlaars and Zhang [13] showed that this point process can be interpreted as a multiple orthogonal polynomial ensemble. The average characteristic polynomials of the matrices $Y_r^*Y_r$ are given as generalized hypergeometric polynomials of the form

$$(-1)^n \prod_{l=1}^{r}(\nu_l+1)_n \, {}_1F_r\left(\begin{matrix} -n \\ \nu_1+1, \quad \ldots, \quad \nu_r+1 \end{matrix} \middle| x \right),$$

where $\nu_j = N_j - N_0$ for $j \in \{1,\ldots,r\}$ (see [13,3]). These polynomials have been studied by the first author [17] with respect to their behavior on the region of zeros (after suitable rescaling) in form of an asymptotic formula of Plancherel–Rotach type. Moreover, using this representation it was shown that the asymptotic zero distribution is given by the Fuss–Catalan distribution of order r (which matches with the known fact that the macroscopic density of the eigenvalues of the matrices $Y_r^*Y_r$ is given by the Fuss–Catalan distribution).

As well as the consideration of products solely consisting of Ginibre matrices it is of interest to study products involving factors with different distributions. This has been done, for instance, by Forrester [6], where products of complex Gaussian and inverse complex Gaussian matrices are studied. In this context it is also interesting to involve Haar distributed unitary factors (see, e.g., [8]). Recently, in [12] the authors considered the squared singular values of products of the type in (1.1) in which the first matrix G_1 is replaced by a truncated unitary random matrix X. More precisely, let $r > 1$, U be a Haar distributed unitary $l \times l$ matrix and let X be the $(n+\nu_1) \times n$ upper left block of U, where $\nu_1 \geq 0$ and $l \geq 2n+\nu_1$. Now let us consider the product of independent matrices

$$Z_r = G_r \cdots G_2 X, \tag{1.2}$$

where G_j are Ginibre matrices of size $(n+\nu_j)\times(n+\nu_{j-1})$, $\nu_j \geq 0$. It is known [12] that the squared singular values of Z_r form a determinantal point process with joint probability distribution on $(0,\infty)^n$ given by a density (with respect to the Lebesgue measure) proportional to

$$\prod_{1\leq j<k\leq n}(x_k - x_j)\det\left\{w_{k-1}(x_j)\right\}_{j,k=1}^{n},$$

where the functions w_k are given as Meijer G-functions by

$$w_k(x) = G_{1,r}^{r,0}\left(\begin{matrix} l-2n+1+k \\ \nu_r, \ldots, \nu_2, \nu_1+k \end{matrix} \middle| x \right).$$

Moreover, the average characteristic polynomials of the Wishart type random matrices $Z_r^*Z_r$ are given by the generalized hypergeometric polynomials [12]

$$(-1)^n \prod_{i=1}^{r}\frac{\Gamma(n+1+\nu_i)}{\Gamma(\nu_i+1)}\frac{\Gamma(l-n+1)}{\Gamma(l+1)} \, {}_2F_r\left(\begin{matrix} -n, \quad l-n+1 \\ \nu_1+1, \quad \ldots, \quad \nu_r+1 \end{matrix} \middle| x \right).$$

In this paper we study the behavior of these polynomials and their zeros for large values of n, where we consider $\nu_j \geq 0$ and $\kappa = l-2n+1 \geq 0$ as fixed integers. This means that the dimensions of the associated matrices grow to infinity in a way described by the fixed differences ν_j of the sizes. Moreover, for the sake of a more convenient analysis, we consider one of the parameters ν_j to be zero, where we (arbitrarily) choose $\nu_r = 0$. Thus, here the average characteristic polynomials are given by

$$\begin{aligned} P_n(x) &= (-1)^n \prod_{i=1}^{r}\frac{\Gamma(n+1+\nu_i)}{\Gamma(\nu_i+1)}\frac{\Gamma(\kappa+n)}{\Gamma(\kappa+2n)} \, {}_2F_r\left(\begin{matrix} -n, \quad n+\kappa \\ \nu_1+1, \quad \ldots, \quad \nu_{r-1}+1, \quad 1 \end{matrix} \middle| x \right) \\ &= (-1)^n n! \prod_{i=1}^{r-1}\Gamma(n+1+\nu_i)\frac{\Gamma(\kappa+n)}{\Gamma(\kappa+2n)}F_n(x), \end{aligned}$$

where we introduce the polynomials F_n by

$$F_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{(n+\kappa)_k(-x)^k}{k!(\nu_1+k)!\cdots(\nu_{r-1}+k)!}. \tag{1.3}$$

In many cases there is a close connection between random matrices and their average characteristic polynomials. For instance, one expects that (after proper rescaling) the limiting distribution of the zeros of the characteristic polynomials coincides with the macroscopic density of eigenvalues (see, e.g., [10]). It will emerge from our analysis that this expectation holds true for the class of Wishart type random matrices $Z_r^*Z_r$ (see (1.2)) under consideration here.

The paper is structured as follows: After stating some auxiliary results in Section 2 we derive the large n behavior of the suitably rescaled polynomials F_n on the region of zeros in form of an asymptotic formula of Plancherel–Rotach type. More