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Equalities for estimators of partial parameters under linear model with restrictions

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ABSTRACT

Estimators of partial parameters in general linear models involve some complicated operations of the submatrices in the given matrices and their generalized inverses in the models. In this case, more efforts are needed to find variety of properties hidden behind these estimators. In this paper, we use some new analytical tools in matrix theory to investigate the connections between the ordinary least-squares estimators and the best linear unbiased estimators of the whole and partial unknown parameters in general linear model with restrictions. In particular, we derive necessary and sufficient conditions for the ordinary least-squares estimators to be the best linear unbiased estimators of the whole and partial unknown parameters in the model.

1. Introduction

Consider a general linear model defined by

$$\mathscr{M}:\ \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad E(\boldsymbol{\varepsilon}) = \mathbf{0},\ D(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma}, \tag{1.1}$$

and a general linear model with parameter restrictions defined by

$$\mathscr{M}_{\rm r}:\ \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \mathbf{A}\boldsymbol{\beta} = \mathbf{b},\ E(\boldsymbol{\varepsilon}) = \mathbf{0},\ D(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma}, \tag{1.2}$$

where $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is a vector of observable random variables, $\mathbf{X} \in \mathbb{R}^{n\times p}$ is a known matrix of arbitrary rank, $\boldsymbol{\beta} \in \mathbb{R}^{p\times 1}$ is a vector of fixed but unknown parameters, $E(\boldsymbol{\varepsilon})$ and $D(\boldsymbol{\varepsilon})$ denote the expectation vector and the dispersion matrix of the random error vector $\boldsymbol{\varepsilon}$, $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is a known non-negative definite matrix of arbitrary rank, $\mathbf{A}\boldsymbol{\beta} = \mathbf{b}$ in (1.2) is a consistent linear matrix equation with $\mathbf{A} \in \mathbb{R}^{m\times p}$ and $\mathbf{b} \in \mathbb{R}^{m\times 1}$. Linear models with parameter restrictions widely in statistical inferences, and are classic issues in regression analysis; see, e.g., [2–5,9,19]. The parameters restriction in (1.2) often occurs, for example, in the linear hypothesis testing on $\boldsymbol{\beta}$ in (1.1).

In statistical analysis, regression models are often written as the sums of some parts of regressors in the model. Through the partitions of regressors in a given linear model, it is quite convenient to determine the contributions of each subset of regressors, and to derive estimators of partial unknown parameters and their functions under such a partitioned regression

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model. For this purpose, we rewrite (1.1) and (1.2) with $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2)$ and $\mathbf{b} = [\,\mathbf{b}_1', \mathbf{b}_2'\,]'$ in the following partitioned forms

$$\mathscr{P}: \ \mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}, \quad E(\boldsymbol{\varepsilon}) = \mathbf{0}, \ D(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma}, \tag{1.3}$$

$$\mathscr{P}_r: \ \mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}, \quad \mathbf{A}_1\boldsymbol{\beta}_1 = \mathbf{b}_1, \ \mathbf{A}_2\boldsymbol{\beta}_2 = \mathbf{b}_2, \ E(\boldsymbol{\varepsilon}) = \mathbf{0}, \ D(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma}, \tag{1.4}$$

where $\mathbf{X}_i \in \mathbb{R}^{n\times p_i}$ with $\mathbf{X} = [\,\mathbf{X}_1, \mathbf{X}_2\,]$, $i = 1, 2$, $\boldsymbol{\beta}_i \in \mathbb{R}^{p_i\times 1}$ with $\boldsymbol{\beta} = [\,\boldsymbol{\beta}_1', \boldsymbol{\beta}_2'\,]'$ and $p = p_1 + p_2$, $i = 1, 2$. Partitioned linear models have been one of the attractive research objects in regression analysis. This kind of models are used in the estimations of partial parameters in regression models, as well as in the investigations of some reduced or small models associated with the original models. Because of the restrictions in (1.4), the derivations and the calculations of estimators of the whole and partial parameter vectors $\boldsymbol{\beta}$, $\boldsymbol{\beta}_1$, and $\boldsymbol{\beta}_2$ in (1.4) involve certain complicated operations of the given matrices and their generalized inverses in (1.4).

Assume that a matrix $\mathbf{K} \in \mathbb{R}^{k\times p}$ is given. Then, the vector $\mathbf{K}\boldsymbol{\beta}$ of parameter functions is said to be estimable under (1.1) if there exists a linear statistic $\mathbf{Ly}$, where $\mathbf{L} \in \mathbb{R}^{k\times n}$, such that $E(\mathbf{Ly}) = \mathbf{LX}\boldsymbol{\beta} = \mathbf{K}\boldsymbol{\beta}$ holds under (1.1). It is well known that

$$\mathbf{K}\boldsymbol{\beta} \text{ is estimable under (1.1)} \Leftrightarrow \mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}') \quad (\text{or equivalently } \mathscr{N}(\mathbf{K}) \supseteq \mathscr{N}(\mathbf{X})); \tag{1.5}$$

see, e.g., [1,24]. As is well known, Ordinary Least Squares Estimators (OLSEs) and Best Linear Unbiased Estimators (BLUEs) of unknown parameters in general linear models deal with the problems of determining estimators of unknown parameters in the models according to certain optimality criteria. These two types of estimator are main issue in the classical theory of linear regression models. Because OLSEs and BLUEs are defined according to two types of optimality criterion, they have different expressions and different algebraic and statistical properties. In such a case, it is of interest to characterize behaviors of these estimators under various assumptions. In particular, it is necessary to compare these estimators and to establish the connections between OLSEs and BLUEs of parametric functions under general linear models due to the simplicity property of OLSEs and optimality property of BLUEs. In fact, OLSEs are equivalent to BLUEs under certain specified assumptions on the given model matrices and dispersion matrices in the models. This kind of problems were widely considered in the statistical literature and various identifying conditions for the equivalence of OLSEs and BLUEs were obtained. For example, Groß et al. [8], Isotalo and Puntanen [10] studied equalities of OLSEs and BLUEs of vectors of unknown parameters under general linear models; Liski et al. [11] established some bounds for the trace of the difference of the covariance matrices of the OLSE and BLUE; Puntanen and Styan [15] studied the connections between the OLSE and BLUE of the mean vector $\mathbf{X}\boldsymbol{\beta}$ in (1.1), presented many equivalent statements for the OLSE and BLUE to be equal, and gave some examples for the OLSE to be the BLUE; Puntanen et al. [16] established some rank equalities for dispersion matrices of OLSEs and BLUEs of parametric functions under general linear models; Tian [23] collected many known and new results on six types of equality of OLSEs and BLUEs for the estimable vector $\mathbf{K}\boldsymbol{\beta}$ under (1.1) and (1.2). OLSEs and BLUEs of partial parameters in general models were also extensively studied, for instance, Qian and Schmidt [17] gave some necessary and sufficient conditions for the OLSE and the BLUE of the partial parameter vector $\boldsymbol{\beta}_1$ in (1.3) to be equal under the assumptions $r(\mathbf{X}) = p$ and $r(\boldsymbol{\Sigma}) = n$; Tian [21,22] and Zhang and Tian [26] characterized the relations between the BLUEs of full and partial parameters under partitioned linear models. Assume that $\mathbf{K}\boldsymbol{\beta}$ is estimable under (1.1) and (1.2), respectively. Then, the four OLSEs and BLUEs of $\mathbf{K}\boldsymbol{\beta}$ under (1.1) and (1.2) are denoted by

$$\mathrm{OLSE}_{\mathscr{M}}(\mathbf{K}\boldsymbol{\beta}), \qquad \mathrm{OLSE}_{\mathscr{M}_r}(\mathbf{K}\boldsymbol{\beta}), \qquad \mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\boldsymbol{\beta}), \qquad \mathrm{BLUE}_{\mathscr{M}_r}(\mathbf{K}\boldsymbol{\beta}).$$

Tian [23] considered the connections among the four estimators, and collected/derived many necessary and sufficient conditions for the following six equalities

$$\mathrm{OLSE}_{\mathscr{M}}(\mathbf{K}\boldsymbol{\beta}) = \mathrm{OLSE}_{\mathscr{M}_r}(\mathbf{K}\boldsymbol{\beta}), \qquad \mathrm{OLSE}_{\mathscr{M}}(\mathbf{K}\boldsymbol{\beta}) = \mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\boldsymbol{\beta}), \qquad \mathrm{OLSE}_{\mathscr{M}}(\mathbf{K}\boldsymbol{\beta}) = \mathrm{BLUE}_{\mathscr{M}_r}(\mathbf{K}\boldsymbol{\beta}),$$
$$\mathrm{OLSE}_{\mathscr{M}_r}(\mathbf{K}\boldsymbol{\beta}) = \mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\boldsymbol{\beta}), \qquad \mathrm{OLSE}_{\mathscr{M}_r}(\mathbf{K}\boldsymbol{\beta}) = \mathrm{BLUE}_{\mathscr{M}_r}(\mathbf{K}\boldsymbol{\beta}), \qquad \mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\boldsymbol{\beta}) = \mathrm{BLUE}_{\mathscr{M}_r}(\mathbf{K}\boldsymbol{\beta})$$

to hold, respectively; for the OLSEs and the BLUEs of the whole and partial mean parameters in (1.3), Tian [25] gave necessary and sufficient conditions for the following two equalities

$$\mathrm{OLSE}_{\mathscr{P}}(\mathbf{X}_1\boldsymbol{\beta}_1) = \mathrm{BLUE}_{\mathscr{P}}(\mathbf{X}_1\boldsymbol{\beta}_1), \qquad \mathrm{OLSE}_{\mathscr{P}}(\mathbf{X}_2\boldsymbol{\beta}_2) = \mathrm{BLUE}_{\mathscr{P}}(\mathbf{X}_2\boldsymbol{\beta}_2)$$

to hold, respectively, and showed the following equivalent statements

$$\mathrm{OLSE}_{\mathscr{P}}(\mathbf{X}\boldsymbol{\beta}) = \mathrm{BLUE}_{\mathscr{P}}(\mathbf{X}\boldsymbol{\beta}) \Leftrightarrow \mathrm{OLSE}_{\mathscr{P}}(\mathbf{X}_i\boldsymbol{\beta}_i) = \mathrm{BLUE}_{\mathscr{P}}(\mathbf{X}_i\boldsymbol{\beta}_i), \quad i = 1, 2.$$

It seems that the above two equivalent statements are neither trivial nor isolated for the OLSEs and the BLUEs under (1.3), and we believe that this kind of connections between OLSEs and BLUEs of whole and partial parameters hold as well under general linear models with parameter restrictions. As a continuation of this work, we consider the following two problems on the connections between the OLSEs and the BLUEs of $\mathbf{X}_i\boldsymbol{\beta}_i$ in (1.4):

(I) Establish necessary and sufficient conditions for the following equalities

$$\mathrm{OLSE}_{\mathscr{P}_r}(\mathbf{X}_1\boldsymbol{\beta}_1) = \mathrm{BLUE}_{\mathscr{P}_r}(\mathbf{X}_1\boldsymbol{\beta}_1), \qquad \mathrm{OLSE}_{\mathscr{P}_r}(\mathbf{X}_2\boldsymbol{\beta}_2) = \mathrm{BLUE}_{\mathscr{P}_r}(\mathbf{X}_2\boldsymbol{\beta}_2) \tag{1.6}$$

to hold under (1.4), respectively.