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Characterization of beta distribution on symmetric cones

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ABSTRACT

In the paper we generalize the following characterization of beta distribution to the symmetric cone setting: let X and Y be independent, non-degenerate random variables with values in $(0, 1)$, then $U = 1 - XY$ and $V = \frac{1-X}{U}$ are independent if and only if there exist positive numbers p_i, $i = 1, 2, 3$, such that X and Y follow beta distributions with parameters $(p_1 + p_3, p_2)$ and (p_3, p_1), respectively.

1. Introduction

In the paper we generalize the following characterization of beta distribution to random matrices and, more generally, to random variables valued in the symmetric cone: *let X and Y be independent, non-degenerate random variables with values in $(0, 1)$, then $U = 1 - XY$ and $V = \frac{1-X}{U}$ are independent if and only if there exist positive numbers p_i, $i = 1, 2, 3$, such that X and Y follow beta distributions with parameters $(p_1 + p_3, p_2)$ and (p_3, p_1), respectively*. This univariate result was proved in [16] under additional assumptions that X and Y have densities, which are strictly positive on $(0, 1)$ and are log-locally integrable. Regularity assumption on densities was removed in the work of [10]. It turns out that the existence of densities assumption is redundant, what was shown in [15].

Here we are interested in a generalization of density versions of the beta characterization, when random variables are valued in the cone Ω_+ of $r \times r$ positive definite symmetric real matrices. Define the analogue of $(0, 1)$ interval in Ω_+: $\mathcal{D}_+ = \{\mathbf{x} \in \Omega_+ : I - \mathbf{x} \in \Omega_+\}$, where I is the identity matrix. Beta distribution on symmetric cone Ω_+ with parameters (p, q) for $p, q > \dim \Omega_+/r - 1$ is defined by its density

$$\mathcal{B}(p, q)(\mathrm{d}\mathbf{x}) = \frac{1}{\mathcal{B}_{\Omega_+}(p, q)} (\det \mathbf{x})^{p-\dim \Omega/r} \det(I - \mathbf{x})^{q-\dim \Omega_+/r} I_{\mathcal{D}_+}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \mathbf{x} \in \Omega_+,$$

where $\mathcal{B}_{\Omega_+(p,q)}$ is the normalizing constant. For any $\mathbf{x} \in \Omega_+$ there exists unique $\mathbf{y} \in \Omega_+$ such that $\mathbf{y}^2 = \mathbf{x}$. Matrix $\mathbf{y}$ is denoted by $\mathbf{y} = \mathbf{x}^{1/2}$. We will show that if X and Y are independent random variables valued in $\mathcal{D}_+$, having continuous densities, which are strictly positive on $\mathcal{D}_+$, then $U = I - X^{1/2} \cdot Y \cdot X^{1/2}$ and $V = U^{-1/2} \cdot (I - X) \cdot U^{-1/2}$ are independent if and only if there exist numbers $p_i > \dim \Omega_+/r - 1$, $i = 1, 2, 3$, such that X and Y follow matrix-variate beta distribution with parameters $(p_1 + p_3, p_2)$ and (p_3, p_1), respectively.

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Actually, we will consider much more general form of transformation of random variables, which is defined through, so-called, multiplication algorithm. A multiplication algorithm is a mapping $w\colon \Omega_+ \mapsto GL(r,\mathbb{R})$ such that $w(\mathbf{x})\cdot w^\top(\mathbf{x}) = \mathbf{x}$ for any $\mathbf{x}\in\Omega_+$, where $GL(r,\mathbb{R})$ is the group of invertible $r\times r$ matrices and $w^\top(\mathbf{x})$ is the transpose of $w(\mathbf{x})$. Multiplication algorithms (actually their inverses called division algorithms) were introduced by Olkin and Rubin [13] alongside the characterization of Wishart probability distribution (see also [3] for generalization to symmetric cone setting). The two basic examples of multiplication algorithms are $w_1(\mathbf{x}) = \mathbf{x}^{1/2}$ ($\mathbf{x}^{1/2}$ being the unique positive definite symmetric square root of $\mathbf{x}$) and $w_2(\mathbf{x}) = t_{\mathbf{x}}$, where $t_{\mathbf{x}}$ is the lower triangular matrix from the Cholesky decomposition of $\mathbf{x} = t_{\mathbf{x}}\cdot t_{\mathbf{x}}^\top$.

We will consider the independence of $U = I - w(X)\cdot Y\cdot w^\top(X)$ and $V = (\widetilde{w}(U))^{-1}\cdot(I-X)\cdot(\widetilde{w}^\top(U))^{-1}$, where w and $\widetilde{w}$ are two multiplication algorithms satisfying additionally some natural conditions. It turns out that, depending on the choice of multiplication algorithms, the characterized distribution may not be the beta distribution (see Theorem 6). For example, when $w=\widetilde{w}=w_2$ the condition of independence of U and V characterizes wider family of distributions called beta-Riesz, which include beta distribution as a special case.

As in the famous Lukacs–Olkin–Rubin Theorem (see [14] for Ω_+ case and [3] for all symmetric cones) the assumption of invariance under the group of automorphisms of distributions of X and Y is considered. The distribution of X is said to be invariant under the group of automorphisms if $O\cdot X\cdot O^\top \overset{d}{=} X$ for any orthogonal matrix O. This approach leads to a characterization of beta distribution regardless of the choice of multiplication algorithms (see Theorem 8).

We cannot give the explicit formula for densities for any multiplication algorithms. In general case, the densities are given in terms of, so-called, w-logarithmic Cauchy functions, that is, functions that satisfy the following functional equation

$$f(\mathbf{x}) + f(w(I)\cdot\mathbf{y}\cdot w^\top(I)) = f(w(\mathbf{x})\cdot\mathbf{y}\cdot w^\top(\mathbf{x})),\quad (\mathbf{x},\mathbf{y})\in\Omega_+.$$

The form of w-logarithmic Cauchy functions without any regularity assumptions for two basic examples of multiplication algorithms were recently considered in [9]. Later on we will write $\mathfrak{w}(\mathbf{x})$ for the linear operator acting on Ω_+ such that $\mathfrak{w}(\mathbf{x})\mathbf{y} = w(\mathbf{x})\cdot\mathbf{y}\cdot w^\top(\mathbf{x})$. $\mathfrak{w}(\mathbf{x})$ will also be termed a multiplication algorithm.

Analogous characterization of Wishart distribution, when densities of respective random variables are given in terms of w-logarithmic functions is given in [8]. Unfortunately, we cannot answer the question whether there exists multiplication algorithm resulting in characterizing other distribution than beta or beta-Riesz. Moreover, the removal of the assumption of the existence of densities remains a challenge.

The idea of the proof is analogous to that of [16]. The independence condition gives us the functional equation for densities, which is then solved. As was observed in [10], in univariate case, the independence condition leads to the generalized fundamental equation of information, that is

$$F(x) + G\left(\tfrac{y}{1-x}\right) = H(y) + K\left(\tfrac{x}{1-y}\right),$$

where $(x,y)\in \mathrm{D}_0 = \left\{(x,y)\in(0,1)^2\colon x+y\in(0,1)\right\}$ and $F,G,H,K\colon (0,1)\to\mathbb{R}$ are unknown functions. Our proof will heavily rely on the solution to the generalization of this equation to the cone Ω_+, which was given in [7].

Similar characterization of beta distribution for random matrices was proved under numerous additional assumptions in [6]. The characterization of 2×2 matrix-variate beta distribution was also given by Bobecka and Wesolowski [2], but the characterization condition was of a different nature.

All above considerations can be generalized to the symmetric cones, of which Ω_+ is the prime example. The paper is organized as follows. In the next section we give necessary introduction to the theory of symmetric cones. Next, in Section 3 we define beta and beta-Riesz probability distributions on symmetric cones. Main theorems are stated and proved in Section 4. Section 5 is devoted to the analysis of the problem, when X and Y have distributions invariant under the group of automorphisms.

2. Preliminaries

In this section we recall basic facts of the theory of symmetric cones, which are needed in the paper. For further details we refer to [4].

A *Euclidean Jordan algebra* is a Euclidean space $\mathbb{E}$ (endowed with scalar product denoted $\langle\mathbf{x},\mathbf{y}\rangle$) equipped with a bilinear mapping (product)

$$\mathbb{E}\times\mathbb{E}\ni(\mathbf{x},\mathbf{y})\mapsto\mathbf{xy}\in\mathbb{E}$$

and a neutral element $\mathbf{e}$ in $\mathbb{E}$ such that for all $\mathbf{x},\mathbf{y},\mathbf{z}$ in $\mathbb{E}$:

(i) $\mathbf{xy}=\mathbf{yx}$,
(ii) $\mathbf{x}(\mathbf{x}^2\mathbf{y}) = \mathbf{x}^2(\mathbf{xy})$,
(iii) $\mathbf{xe}=\mathbf{x}$,
(iv) $\langle\mathbf{x},\mathbf{yz}\rangle = \langle\mathbf{xy},\mathbf{z}\rangle$.