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Tail asymptotics for the bivariate skew normal

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ABSTRACT

We derive the asymptotic rate of decay to zero of the tail dependence of the bivariate skew normal distribution under the equal-skewness condition $\alpha_1 = \alpha_2, = \alpha$, say. The rate depends on whether $\alpha > 0$ or $\alpha < 0$. For the lower tail, the latter case has rate asymptotically identical with the bivariate normal ($\alpha = 0$), but has a different multiplicative constant. The case $\alpha > 0$ gives a rate dependent on α. The detailed asymptotic behaviour of the quantile function for the univariate skew normal is a key. This study is partly a sequel to our earlier one on the analogous situation for bivariate skew t.

1. Introduction

The coefficient of lower tail dependence of a random vector $\mathbf{X} = (X_1, X_2)^\top$ with marginal inverse distribution functions F_1^{-1} and F_2^{-1} is defined as

$$\lambda_L = \lim_{u \to 0^+} \lambda_L(u), \quad \text{where } \lambda_L(u) = P(X_1 \le F_1^{-1}(u) | X_2 \le F_2^{-1}(u)). \tag{1}$$

$\mathbf{X}$ is said to have asymptotic lower tail dependence if λ_L exists and is positive. If $\lambda_L = 0$, then $\mathbf{X}$ is said to be asymptotically independent in the lower tail. This quantity provides insight on the tendency for the distribution to generate joint extreme event since it measures the strength of dependence (or association) in the lower tails of a bivariate distribution. If the marginal distributions of these random variables are continuous, then from (1), it follows that $\lambda_L(u)$ can be expressed in terms of the copula of $\mathbf{X}$, $C(u_1, u_2)$, as

$$\lambda_L(u) = \frac{P(X_1 \le F_1^{-1}(u), X_2 \le F_2^{-1}(u))}{P(X_2 \le F_2^{-1}(u))} = \frac{C(u, u)}{u}. \tag{2}$$

The foci of such studies in asymptotic dependence/independence are therefore the asymptotic behaviour as $u \to 0^+$ of $F_i^{-1}(u)$, $i = 1, 2$ and of $C(u, u) = P(X_1 \le F_1^{-1}(u), X_2 \le F_2^{-1}(u))$.

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In the case of standard bivariate normal with correlation coefficient ρ, $-1 < \rho < 1$, where the distribution function for each univariate marginal is denoted by $\Phi(\cdot)$, the asymptotic behaviour is relatively well-known:

$$\Phi^{-1}(u) \sim -\sqrt{-2\log(u\sqrt{-4\pi\log u})} \tag{3}$$

$$C(u,u) \sim c(\rho)u^{\frac{2}{1+\rho}}(-\log u)^{\frac{-\rho}{1+\rho}} \tag{4}$$

as $u \to 0^+$, where $c(\rho) = (1+\rho)^{3/2}(1-\rho)^{-1/2}(4\pi)^{-\rho/(1+\rho)}$. The expression (4) maybe found (for upper tail dependence) in Reiss and Thomas [24, p. 322], where it is attributed to Ledford and Tawn [15]. A sketch proof in a more general setting may be found in Ledford and Tawn [16, Appendix A]: Bivariate Normal Results, which also exhibits (3). A proof of (3) and (4) is also contained in Fung and Seneta [11], whose methodology we shall use below.

Ramos and Ledford [23], continuing the work of Ledford and Tawn [16], studied intensively a family of bivariate distributions (which they characterised) which satisfied in particular the condition

$$C(u,u) = u^{\frac{1}{\omega}}L(u) \tag{5}$$

where $L(u)$ is a slowly varying function (SVF) as $u \to 0^+$, and $\omega \in (0, 1]$, so that, in fact, the value of ω could be used for comparison of the degree of tail dependence structure between members of the family. The standard bivariate extreme value models correspond to $\omega = 1$; and independence copula models correspond to $\omega = 1/2$.

Both Hashorva [13] and Hua and Joe [14] have also pursued this idea. In particular they allude to residual and intermediate tail dependence respectively. The definition of Hua and Joe [14], which is more consistent with our theoretical development, will be adopted here. This paper defines $\kappa = 1/\omega$ in (5) as the (lower) tail order of a copula. The tail order case $1 < \kappa < 2$ is considered as intermediate tail dependence as it corresponds to the copula having some level of positive dependence in the tail when $\lambda_L = 0$. Thus when $\lambda_L(u) = C(u,u)/u = u^{\kappa-1}L(u)$, $1 < \kappa < 2$, there is some measure of positive association when $\lambda_L = 0$, but the association is not as strong as when $\kappa = 1$, and $\lambda_L(u) = L(u) \to \lambda_L > 0$, $u \to 0^+$, the case of asymptotic tail dependence.

The standard bivariate normal with correlation coefficient $-1 < \rho < 1$ corresponds to $\omega = \frac{1+\rho}{2}$ in (5), and hence $\rho > 0$ is an instance of intermediate tail dependence.

The motivating idea of building an extension of the normal class of distributions by introducing skewness dates back to a paper by Azzalini [1], and has developed into an extensive theory presented in a recent monograph by Azzalini and Capitanio [3].

The bivariate skew normal distribution was introduced in Azzalini and Dalla Valle [4] (which is discussed further in Azzalini and Capitanio [2]); Azzalini and Capitanio [3] contains a review. A random vector $\mathbf{X}$ is said to have a bivariate skew normal distribution, denoted as $\mathbf{X} \sim SN_2(\boldsymbol{\alpha}, R)$, if the probability density of $\mathbf{X}$ is

$$f(\mathbf{x}) = 2\phi_2(\mathbf{x}, R)\Phi(\boldsymbol{\alpha}^\top \mathbf{x}), \tag{6}$$

where $\phi_2(\cdot, R)$ is density of a bivariate normal distribution with mean $\mathbf{0}$ and correlation matrix R and $\Phi(\cdot)$ is the cdf of a univariate standard normal distribution. The correlation matrix R and skew vector $\boldsymbol{\alpha}$ are defined as $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, with $-1 < \rho < 1$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^\top \in \mathbb{R}^2$ respectively. Obviously, the (symmetric) bivariate normal is obtained as special case when $\boldsymbol{\alpha} = 0$.

The results of Lysenko, Roy and Waeber [19] and Bortot [5] show that the skew normal distribution is tail independent, that is: $\lambda_L = \lim_{u\to 0^+} \lambda_L(u) = 0$.

The primary focus of this current note is thus to consider (5) in the setting of the bivariate skew normal distribution, and specifically to see how the introduction of skewness into the bivariate normal distribution affects (3), and (4), with the same detail of specific expression of the regularly varying function. We find that the regularly varying index κ depends on whether $\alpha > 0$ or $\alpha < 0$, and these two cases of α require quite different treatments. The case $\alpha < 0$ is asymptotically (apart from a constant multiplier) identical to the symmetric bivariate normal case $\alpha = 0$.

We mention that the treatments of Hashorva [13] and Hua and Joe [14] focus on a general multivariate situation, with specific attention to the value of $\kappa = 1/\omega$ in (5), and to the limit behaviour of $L(u)$, $u \to 0^+$, whereas we provide an explicit asymptotic form for the function $L(\cdot)$ in (5) for the specific instance of bivariate skew normal in each α case, in parallel with the expression (4) for the standard bivariate normal. This adds to the understanding of skew normal theory already extensively exposited in Azzalini and Capitanio [3], and to the understanding of the effect of skewing in general.

While the skew normal is tail independent, its various extensions, see Fung and Seneta [10], Bortot [5] and Padoan [22] for the bivariate skew t; and Ling and Peng [17] for the bivariate skew slash, are tail dependent. This motivated the ideology of Fung and Seneta [12] which was published in this same journal and subsequently our analysis of the bivariate skew normal. The skew t distribution studied there is mixing on the bivariate skew normal, inasmuch as $\mathbf{X}$ has this distribution if $\mathbf{X} \sim V^{-1/2}\mathbf{Z}$ where $\mathbf{Z} \sim SN_2(\boldsymbol{\alpha}, R)$ where independently $V \sim \Gamma(\eta/2, \eta/2)$. This distribution is tail dependent i.e. $\lambda_L > 0$ and Fung and Seneta [12] obtained for it the rate of convergence result:

$$\lambda_L(u) - \lambda_L = \mathcal{K}(\eta, R, \boldsymbol{\alpha})u^{\frac{2}{\eta}} + O\left(u^{\frac{4}{\eta}}\right) \tag{7}$$