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Elliptical affine shape distributions for real normed division algebras

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HIGHLIGHTS

- Statistical affine shape theory in the context of real normed division algebras is studied.
- Affine shape density is obtained.
- The complex normal affine density is derived and applied in brain magnetic resonance scans of normal and schizophrenic patients.

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ABSTRACT

The statistical affine shape theory is set in this work in the context of real normed division algebras. The general densities apply for every field: real, complex, quaternion, octonion, and for any noncentral and non-isotropic elliptical distribution; then the separated published works about real and complex shape distributions can be obtained as corollaries by a suitable selection of the field parameter and univariate integrals involving the generator elliptical function. As a particular case, the complex normal affine density is derived and applied in brain magnetic resonance scans of normal and schizophrenic patients.

1. Introduction

The literature of random matrix distributions tells us about a great effort for obtaining separately results on real, complex, quaternion and octonion fields, that were recently derived under a general approach, see for example, [36,21,35,9,12,11].

In fact, many models and techniques are explored first in the real case, and then the complex case is proposed with all the necessary mathematical tools. In the last 60 years we can quote dozens of these extensions, see for example [27,30,38,32,8,41], among many others.

However, during the past 30 years a unified theory of random matrix distributions has reached a substantial development. Essentially, these advances have been achieved through two approaches based on the theory of Jordan algebras and the real

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normed division algebras. A basic source of the theory of random matrix distributions under Jordan algebras can be found in [20,36,7,25,26], and the references therein. Parallel results on theory of random matrix distributions based on real normed division algebras are given by Gross and Richards [23], Dumitriu [16], Forrester [21], Díaz-García and Gutiérrez-Jáimez [10], Díaz-García and Gutiérrez-Jáimez [12], among others. Moreover, new connections with abstract algebra were studied by Ishi [29,4]; instead of Jordan algebras they considered *normal j-algebras* and *Vinberg algebras*, respectively.

Statistical shape theory followed a similar way of evolution; initial results were obtained in the classical real case based on Gaussian models and then some translations to the complex case were published. A summary of the real statistical shape theory can be found in [15] and the references therein. The complex case has few results, see for example [37,15].

There are several geometrical approaches in shape theory (Euclidean, affine, projective, Eulerian, etc.), we will focus in the affine case. Goodall and Mardia [22] (corrected by Díaz-García et al. [13,6]) proposed an alternative system shape coordinates termed configuration or affine coordinates, randomly indexed by a matrix multivariate gaussian distribution. Then, Caro-Lopera et al. [6] extended this theory by replacing the normality assumption with a matrix multivariate elliptical law. Those works were studied in the real field, so if we follow the tradition of the cited literature, we can expect an extension to the complex case, for example, by studying the new Jacobians, integrals and computations.

Instead of that procedure, we propose a unified approach for the affine statistical theory of shape, by studying the real, complex, quaternion and octonion cases in a simultaneous way. As we shall see in Section 2, these four cases are formally termed: real normed division algebras. However, as usual, this type of generalisation involves some new definitions and concepts; next we summarise some elements of abstract algebra.

We start with a special field. From an applied point of view, the relevance of *the octonions* remains unclear. An excellent review of the history, construction and many other properties of octonions is given in [1], where it is stated that:

> "*Their relevance to geometry was quite obscure until 1925, when Élie Cartan described 'triality' — the symmetry between vector and spinors in 8-dimensional Euclidean space. Their potential relevance to physics was noticed in a 1934 paper by Jordan, von Neumann and Wigner on the foundations of quantum mechanics… Work along these lines continued quite slowly until the 1980s, when it was realised that the octionions explain some curious features of string theory…* ***However, there is still no proof that the octonions are useful for understanding the real world****. We can only hope that eventually this question will be settled one way or another*".

For the sake of completeness, the octonions will be considered in this work, but we must recognise that the application of the associated results can only be conjectured. Even so, some expectations are emerging, for example, [21, Section 1.4.5, pp. 22-24] proved that the bi-dimensional eigenvalue density function of a 2×2 octonionic matrix Gaussian ensemble is obtained from the eigenvalue general joint density function of a Gaussian ensemble with $m = 2$ and $\beta = 8$, see notation in Section 2. Moreover, according to Faraut and Korányi [20] and Sawyer [42], it is easy to check that the results of this work are valid for the *algebra of Albert*, i.e., when the involved hermitian matrices or certain products of hermitian matrices are 3×3 octonionic matrices.

The present paper is organised as follows: basic concepts of abstract algebra, integral properties of Jack polynomials and generalised hypergeometric functions are given in Section 2; then, a Jacobian with respect to Lebesgue measure for real normed division algebras is obtained in Section 3 and the main results of the paper are derived as a consequence; in Section 4, the affine shape distribution for several particular elliptical laws are derived; and finally, an application from the literature of shape in the complex case is studied in Section 5.

2. Preliminary results

A discussion of real normed division algebras can be found in [1,39]; meanwhile, theory of Jack polynomials and hypergeometric functions can be read in [42,23,20,33,9]. For convenience, we shall introduce some notations, although in general we adhere to standard notations.

Let $\mathbb{F}$ be a field. An *algebra* $\mathfrak{F}$ over $\mathbb{F}$ is a pair $(\mathfrak{F}; m)$, where $\mathfrak{F}$ is a *finite-dimensional vector space* over $\mathbb{F}$ and *multiplication* $m : \mathfrak{F} \times \mathfrak{F} \to A$ is an $\mathbb{F}$-bilinear map; that is, for all $\lambda \in \mathbb{F},\ x, y, z \in \mathfrak{F}$,

$$m(x, \lambda y + z) = \lambda m(x; y) + m(x; z)$$
$$m(\lambda x + y; z) = \lambda m(x; z) + m(y; z).$$

Two algebras $(\mathfrak{F}; m)$ and $(\mathfrak{E}; n)$ over $\mathbb{F}$ are said to be *isomorphic* if there is an invertible map $\phi : \mathfrak{F} \to \mathfrak{E}$ such that for all $x, y \in \mathfrak{F}$,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

For notational simplicity, we write $m(x; y) = xy$ for all $x, y \in \mathfrak{F}$.

Let $\mathfrak{F}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{F}$ is said to be:

1. *alternative*, if $x(xy) = (xx)y$ and $x(yy) = (xy)y$ for all $x, y \in \mathfrak{F}$;
2. *associative*, if $x(yz) = (xy)z$ for all $x, y, z \in \mathfrak{F}$;
3. *commutative*, if $xy = yx$ for all $x, y \in \mathfrak{F}$; and
4. *unital*, if there is a $1 \in \mathfrak{F}$ such that $x1 = x = 1x$ for all $x \in \mathfrak{F}$.

If $\mathfrak{F}$ is unital, then the identity 1 is uniquely determined.