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Bivariate one-sample optimal location test for spherical stable densities by Pade' methods

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ABSTRACT

Complex signal detection in additive noise can be performed by a one-sample bivariate location test. Spherical symmetry is assumed for the noise density as well as closedness with respect to linear transformation. Therefore the noise is assumed to have spherical distribution with α-stable radial density. In order to cope with this difficult setting the original sample is transformed by Pade' methods giving rise to a new sample with universality properties. The stability assumption is then reduced to the Gaussian one and it is proved that a known van der Waerden type test, with optimal properties, based on the new sample can be used. Furthermore a new test is proposed whose asymptotic relative efficiency w.r. to the van der Waerden type test when applied to the new sample is larger than one.

0. Introduction

Additive noise filtering is a common problem in many experimental situations. Sometimes happens however that what matters is to understand if a signal is present or not in the observations. The specific shape of the signal is not relevant. Moreover sometimes it is not possible to make assumptions on the statistical distribution of the noise. The problem consists then in characterizing the noise w.r. to any possible signal with the only constraint that the noise is additive. In the following we assume that the noise can be represented by a discrete time, complex valued, stationary process such that every finite set of random variables of the process has a multivariate spherical distribution centered in zero. In the limit case in which this process reduces to a single complex random variable this is equivalent to considering a couple of real random variables with bivariate spherical distribution. As the noise is additive when a signal is present the data have the same multivariate spherical distribution centered on the signal. We remember that this assumption generalizes to dimension larger than one the natural assumption that the noise should have a symmetric distribution w.r. to zero i.e. negative values have the same distribution as positive ones. Exchangeability is implied by sphericity, therefore every finite set of random variables of the noise process has the same distribution of a permutation of its components. This seems a natural requirement for the noise. However sphericity implies more geometric structure. In fact e.g. the distribution of every n-dimensional set of random variables of a spherical process is invariant by rotation in the n-dimensional Euclidean space of random variables. This too seems a natural property of the noise. Moreover adding more noise should not modify its statistical distribution. Therefore the radial density of the multivariate spherical distribution should be an α-stable density.

An obvious tool for solving the signal detection problem described above consists in performing a one-sample multivariate location test H_0: the observed process is centered in zero, against H_1: the observed process is not centered in zero. Many methods exist to solve this problem. In the specific case when the noise distribution is Gaussian, Hotelling's test is the optimal one [13]. Otherwise nonparametric tests have been developed which can be classified into three main groups.

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To the first group belong methods based on marginal signs and ranks [20] but are not invariant w.r. to affine transformations. The second group is related with spatial signs and ranks [15,12,11], and [17] for a review. The tests in this group are affine invariant. Methods in the third group are affine invariant and are based on the concept of interdirections [21,19]. In [9] it was proved that it is possible to devise tests, based on interdirections and ranks of the length of the residual vectors, that are affine invariant and exhibit local asymptotic optimality a la Le Cam if the joint density of the observations is elliptically symmetric and the radial density satisfies some assumptions. Unfortunately these assumptions under the α-stability hypothesis are not valid but in some specific cases. The idea is then to transform the original sample in order to be able to make this check [2].

More specifically, given m complex i.i.d. n-variate observations, each of them can be replaced by a transformed one by considering Padé approximants of orders $[n/2-1, n/2]$ if n is even and of orders $[(n-1)/2, (n-1)/2]$ if n is odd of the Z-transform of the n components. Four statistics are then computed: poles, zeros, normalized residuals at the poles and normalized residuals at the zeros in the following called Padé parameters. It turns out that all these quantities are functions of the generalized eigenvalues and eigenvectors of two pencils of random Hankel matrices. It is proved that, under H_0, these statistics are universal, i.e. their distribution does not depend on the specific spherical distribution of the observations. Therefore the α-stable radial density can be replaced by a Gaussian one. Moreover, in the specific case of $n=2$, it is proved that the pole statistic satisfies the hypotheses required in [9]. A van der Waerden type optimal test can then be used on this parameter. A Monte Carlo experiment shows that the same test applied to the other Padé parameters has lower power and the same is true for the same test applied to the original data and for the Hotelling test applied to the original data. Therefore it seems useless to check the hypotheses required in [9] for the other Padé parameters and the original data. We notice that the Chernoff and Savage's result [9, Th. 6] comparing the van der Waerden type test and the Hotelling one on the original data does not hold in general for α-stable data.

Finally it is proved that the poles statistic can be used to define a new optimal test a la Le Cam, whose asymptotic relative efficiency (ARE) w.r. to the van der Waerden type test when applied to poles data is larger than one.

A Monte Carlo experiment confirms these results stressing that the advantages of the new test applied to the poles data is larger for small values of α and signal-to-noise ratio (SNR).

The paper is organized as follows. In Section 1 the statistics are defined and their universality properties are assessed. In Section 2 the sphericity of the statistics is proved and the location tests are described. In Section 3 some simulation results are reported and in the last one two examples of application of the proposed statistic are considered and the results are compared to that reported in the literature for several location tests applied to the same datasets.

1. Universality properties

Let us denote random quantities by bold characters and assume that the complex-valued discrete process $\{\mathbf{a}\}_k,\ k\in\mathbb{N}$, representing the signal plus white noise, is such that all finite sets of $\{\mathbf{a}\}_k$ have an elliptically symmetric distribution. More precisely, if $\underline{\mathbf{a}}=[\mathbf{a}_0,\dots,\mathbf{a}_{n-1}]$, we assume that $\forall n=2p,\ \tilde{\underline{\mathbf{a}}}=\{\Re(\underline{\mathbf{a}}),\Im(\underline{\mathbf{a}})\}$ has an elliptical distribution with a density given by (see e.g. [7])

$$G(\underline{a};\underline{s},\Sigma,g)=\Gamma(n)\|\underline{a}\|_{\underline{s},\Sigma}^{1-2n}g(\|\underline{a}\|_{\underline{s},\Sigma})/(2\pi^n) \tag{1}$$

where $g(\cdot)$ is the density of $\|\underline{\mathbf{a}}\|_{\underline{s},\Sigma}$ and

$$\|\underline{a}\|_{\underline{s},\Sigma}=\left\{(\underline{a}-\underline{s})^\top\Sigma^{-1}(\underline{a}-\underline{s})\right\}^{1/2},\quad \underline{s}\in\mathbb{R}^{2n},\ \Sigma>0\in\mathbb{R}^{2n\times 2n}.$$

Equivalently we can assume that

$$\tilde{\underline{\mathbf{a}}}\overset{d}{=}\underline{s}+\Sigma^{1/2}\underline{\mathbf{e}}$$

where $\underline{s}$ represents the signal and $\underline{\mathbf{e}}$ represents the scaled noise centered in zero with spherical distribution.

The Z-transform of $\{\mathbf{a}_k\}$ is the formal random power series

$$\mathbf{F}(z)=\sum_{k=0}^{\infty}\mathbf{a}_k z^{-k},\quad |z|>1$$

which can be extended to the unit disk by analytic continuation. Let us denote by $[\mathbf{p-1},\mathbf{p}](z)$ the random Padé approximant of $\mathbf{F}(z)$ of order $(p-1,p)$. Its poles are denoted by $\{\boldsymbol{\xi}_j\}, j=1,\dots,p$ and its zeros by $\{\boldsymbol{\zeta}_j\}, j=1,\dots,p-1$. The poles can be computed by noting that (see e.g. [3]) they are the generalized eigenvalues of a pencil of square random Hankel matrices $\mathbf{U}=[\mathbf{U}_1,\mathbf{U}_0]$ where

$$\mathbf{U}_0=\begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_{p-1}\\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p\\ . & . & \cdots & .\\ \mathbf{a}_{p-1} & \mathbf{a}_p & \cdots & \mathbf{a}_{2p-2}\end{bmatrix},\qquad \mathbf{U}_1=\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p\\ \mathbf{a}_2 & \mathbf{a}_3 & \cdots & \mathbf{a}_{p+1}\\ . & . & \cdots & .\\ \mathbf{a}_p & \mathbf{a}_{p+1} & \cdots & \mathbf{a}_{2p-1}\end{bmatrix}.$$