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On the worst and least possible asymptotic dependence

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ABSTRACT

Multivariate extremes behave very differently under asymptotic dependence as compared to asymptotic independence. In the bivariate setting, we are able to characterise the extreme behaviour of the asymptotic dependent case by using the concept of the copula. As a result, we are able to identify the properties of the boundary cases, that are asymptotic independent but still have some asymptotic dependent features. These situations are the most problematic in statistical extreme, and, for this reason, distinguishing between asymptotic dependence and asymptotic independence represents a difficult problem. We propose a simple test to resolve this issue which is an alternative to the procedure based on the classical coefficient of tail dependence. In addition, we are able to identify the worst/least asymptotic dependence (in the presence of asymptotic dependence) that maximises/minimises the probability of a given extreme region if tail dependence parameter is fixed. It is found that the perfect extreme association is not the worst asymptotic dependence, which is consistent with the existing literature. We are able to find lower and upper bounds for some risk measures of functions of random variables. A particular example is the sum of random variables, for which a vivid academic effort has been noticed in the last decade, where bounds for a sum of random variables are sought. It is numerically shown that our approach provides a great improvement of the existing methods, which reiterates the sensible conclusion that any additional piece of information on dependence would help to reduce the spread of these bounds.

1. Introduction

Estimation of multivariate extreme events is a challenging problem in *Extreme Value Theory* (EVT) and the starting point of non-parametric estimation is to decide if data exhibit the *asymptotic dependence* (AD) or *asymptotic independence* (AI) property. In simple words, under AD, concomitant extreme events are observed and both are at the same scale. Under AI, concomitant extreme events may occur but at different scales or may not even occur at the same time. Therefore, it is expected that extreme regions estimates to be very different in magnitude in the presence of AD than AI. It is well-known that statistical inferences in the presence of AI is very difficult, and many estimation methods are available if AD holds (see for example, [8]). Since distinguishing between AD and AI plays an important role in predicting extreme events, Ledford and Tawn [24,25] introduced the *coefficient of tail dependence* which has been extensively investigated in the literature. For example, nonparametric inference can be found in [27,9], while Goegebeur and Guillou [18] considered an asymptotically unbiased estimator in the case of AI. The main disadvantage of the coefficient of tail dependence is that inconclusive results are possible, especially in situations which fall on the boundary between AD and AI. In order to help detect AI/AD, the recent

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paper of Asimit et al. [3] proposes a conditional version of the classical measure of association Kendall's tau for absolutely continuous distributions.

The initial motivation of the paper was to examine in great detail the joint tail behaviour of a bivariate random vector under AD and understand the differences between AD and almost AD (boundary between AD and AI) cases. Since we are interested in characterising the association of extreme events, the concept of the *copula* will be considered throughout this paper. Our properties will clarify the existing examples in the literature that pointed out naive conjectures of a link between some measure of tail dependence and the presence of AI/AD. Having in mind our AD characterisation, one may construct counterexample for such speculative conclusions and serve to provide a better understanding of extreme behaviour in the almost AD extreme behaviour. In fact, we exhibit one example, but many examples can be constructed in the same fashion, that can be useful as a model for any statistical extreme where the overlapping between AD and AI is of interest. We are able to identify the worst/least extreme dependence under AD with a fixed *tail dependence parameter*, which is a measure of tail dependence (for a summary of tail dependence concepts, we refer the reader to [20]). In our interpretation, worst (least) extreme dependence represents the least (most) favourable dependence that may occur and it really depends on the context. For example, when one deals with a sum of positive insurance losses, the worst (least) dependence is achieved when some tail risk measures of the aggregate risk is maximised (minimised). Note that focusing only on the tail dependence parameter, the overall dependence may be underestimated as argued in [17]. We can further find the upper and lower bounds for the tail distribution of a function of random variables (rv's). A special case is the sum of rv's that has been extensively studied in the literature as it can be seen below. Note that extreme quantile for a sum of rv's are of great interest in risk management among other areas (for example, see [3]).

Value-at-Risk (which is in fact a quantile) is one of the most common risk measure used in practice in the banking and insurance industries, and therefore its evaluation has received particular attention in the last decade. The uncertainty with the dependence among rv's is huge, especially due to the data scarcity, and the choice of a parametric model is quite challenging even though such compromises are made in practice and are sometimes based on prior beliefs of the modeller. As a result, evaluating the range of values for the VaR of a sum of rv's is usually made when the marginal distributions are known and, possibly, an additional piece of information about dependence is known. This approach allows the decision-maker to understand the worst and least possible VaR-based risk. The best possible bounds for the distribution of a sum of rv's are described in [13,14] and the references therein. VaR bounds have been discussed in [15,32,4,5], if no additional information about dependence is available. The recent paper of Bernard et al. [6] investigates the VaR constrained set-up under an additional assumption that the aggregate variance is known. The same problem is investigated in [4,5] when the decision-maker has only a summary statistics of the individual risks (mean, variance, skewness etc., i.e. some high order expectations) instead of their distributions. Usually, these bounds are attained under extreme atomic dependence models which suggests that studying the constrained problem under a reduced set of feasible dependence structures represents the way forward in this field. As a result, Bignozzi et al. [7] find VaR bounds under the assumption of lower orthant stochastic ordering with respect to a particular dependence model.

This paper first provides the necessary background in Section 2. The AD is fully characterised in Section 3, which enables us to identify the worst and least asymptotic dependence in Section 4. We propose a new procedure to identify the presence of AD/AI in Section 5. Section 6 numerically illustrates the advantages of our findings over the existing bounds available in the literature. Finally, all proofs are relegated in the Appendix.

2. Background

Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) rv's with *cumulative distribution function* (cdf) F and infinite right-end point. EVT assumes that there are two sequences of constants $a_n > 0$, $b_n \in \Re$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(a_n\left(\max_{1\le i\le n} X_i - b_n\right) \le x\right) = G(x), \quad x \in \Re.$$

In this case, G is called an *Extreme Value Distribution* and F is said to belong to the *domain of attraction of* G. The Fisher–Tippett Theorem (see [16]) states that if the limit distribution is non-degenerate then $G(x) = \exp\{-x^{-\alpha}\}$ for all $x > 0$ with $\alpha > 0$ or $G(x) = \exp\left\{-e^{-x}\right\}$ for all $x \in \Re$, since the domain of F is assumed to be unbounded in the right tail. In the first case, X has the *regularly varying (RV)* property at ∞ with tail index α, i.e. the survival function $\bar{F} = 1 - F$ satisfies $\lim_{t\to\infty} \bar{F}(tx)/\bar{F}(t) = x^{-\alpha}$ for all $x > 0$, and we write $\bar{F} \in \mathcal{RV}_{-\alpha}$. In the second case, X has a Gumbel tail and it is well-known (see, for example, [29] or [11]) that there exists a positive, measurable function a such that $\lim_{t\to\infty} \bar{F}(t + xa(t))/\bar{F}(t) = e^{-x}$ for all real x, and we write $\bar{F} \in \Lambda(a)$.

We now review the concept of *vague convergence*. Consider an n-dimensional cone $\mathcal{E}$ equipped with a Borel sigma-field $\mathcal{B}$. Two particular cones $\mathcal{E}_{\mathcal{F}} = [\mathbf{0}, \infty] \setminus \{\mathbf{0}\}$ and $\mathcal{E}_{\mathcal{G}} = [-\infty, \infty] \setminus \{-\infty\}$ will be of interest in this paper. In particular, $\mathcal{E}_{\mathcal{F}}$ is involved when the tails are RV, while $\mathcal{E}_{\mathcal{G}}$ becomes the main interest whenever we deal with Gumbel tails. A measure on the cone is called Radon if its value is finite for every compact set in $\mathcal{B}$. For a sequence of Radon measures $\{\nu, \nu_k,\ k = 1, 2, \ldots\}$ on $\mathcal{E}$, we say that ν_k vaguely converges to ν, written as $\nu_k \xrightarrow{v} \nu$, if

$$\lim_{k\to\infty} \int_{\mathcal{E}} h(\mathbf{z})\nu_k(\mathrm{d}\mathbf{z}) = \int_{\mathcal{E}} h(\mathbf{z})\nu(\mathrm{d}\mathbf{z})$$