Journal of Multivariate Analysis 141 (2015) 22–34

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

The Matsumoto–Yor property on trees for matrix variates of different dimensions

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ARTICLE INFO

Article history:
Received 20 March 2014
Available online 19 June 2015

AMS subject classifications:
62E10
60E05
62H05

Keywords:
Wishart distribution
MGIG distribution
Independence
Matsumoto–Yor property
Characterization
Tree
Quasi-Wishart

ABSTRACT

The paper is devoted to an extension of the multivariate Matsumoto–Yor (MY) independence property with respect to a tree with p vertices to the case where random variables corresponding to the vertices of the tree are replaced by random matrices. The converse of the p-variate MY property, which characterizes the product of one gamma and $p-1$ generalized inverse Gaussian distributions, is extended to characterize the product of the Wishart and $p-1$ matrix generalized inverse Gaussian distributions.

1. Introduction

Let $\mathcal{V}_n$ be the Euclidean space of $n \times n$ real symmetric matrices equipped with the inner product $\langle a, b\rangle = \mathrm{trace}(ab)$. Let dx denote the Lebesgue measure on $\mathcal{V}_n$ assigning the unit mass to the unit cube. Let $\mathcal{V}_n^+$ denote the cone of positive definite matrices in $\mathcal{V}_n$ and let $\overline{\mathcal{V}}_n^+$ denote its closure. For $x \in \mathcal{V}_n$ let $|x|$ denote the determinant of x.

Let $c \in \mathcal{V}_n^+$ and $q \in \Lambda_n = \{0, \frac{1}{2}, \frac{2}{2}, \dots, \frac{n-1}{2}\} \cup (\frac{n-1}{2}, \infty)$. The random matrix Y taking its values in $\overline{\mathcal{V}}_n^+$ is said to follow the Wishart $W_n(q, c)$ distribution if its Laplace transform is given by

$$L_Y(\theta) = \frac{|c|^q}{|c-\theta|^q}, \quad c - \theta \in \mathcal{V}_n^+,$$

see Casalis and Letac [4] and references given therein. When $q > \frac{n-1}{2}$, that is when Y takes its values in $\mathcal{V}_n^+$, this distribution has density of the form

$$f_Y(y) = \frac{|c|^q}{\Gamma_n(q)} |y|^{q-\frac{n+1}{2}} \exp\left(-\langle c, y\rangle\right) \mathbf{1}_{\mathcal{V}_n^+}(y),$$

where Γ_n denotes the multivariate gamma function, see Muirhead [19]. When $q \in \Lambda_n$ and $q \le \frac{n-1}{2}$ the distribution is singular and is concentrated on the boundary of $\bar{\mathcal{V}}_n^+$. In the special case $q = 0$, it is the Dirac measure concentrated at the zero matrix.

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http://dx.doi.org/10.1016/j.jmva.2015.05.018

A random matrix X, taking its values in $\mathcal{V}_n^+$, is said to follow the matrix generalized inverse Gaussian distribution, $MGIG_n(-p, a, b)$, if it has density of the form

$$f_X(x) = \frac{1}{K_p^{(n)}(a, b)} |x|^{-p-\frac{n+1}{2}} \exp\left(-\langle a, x\rangle - \left\langle b, x^{-1}\right\rangle\right) \mathbf{1}_{\mathcal{V}_n^+}(x), \tag{1.1}$$

where $K_p^{(n)}$ is the matrix variate modified Bessel function of the third kind, see Herz [6].

Letac [12] has observed that the $MGIG_n(-p, a, b)$ is well defined iff p, a, b satisfy one of the following three conditions:

1. $a, b \in \mathcal{V}_n^+$ and $p \in \mathbb{R}$,
2. $a \in \bar{\mathcal{V}}_n^+$ with $\mathrm{rank}(a) = m \in \{0, 1, \dots, n-1\}$, $b \in \mathcal{V}_n^+$ and $p > \frac{n-m-1}{2}$,
3. $a \in \mathcal{V}_n^+$, $b \in \bar{\mathcal{V}}_n^+$ with $\mathrm{rank}(b) = m \in \{0, 1, \dots, n-1\}$ and $p < -\frac{n-m-1}{2}$.

This extends earlier definitions of the matrix variate GIG as given in Bardorff-Nielsen et al. [1] or Butler [3].

The *MGIG* distribution has the following property, which will be used later on

$$\text{if } X \sim MGIG_n(-p, a, b) \text{ then } X^{-1} \sim MGIG_n(p, b, a). \tag{1.2}$$

There are several connections between the Wishart and *MGIG* distributions considered in the literature, see e.g. Bardorff-Nielsen and Koudou [2], Butler [3], Koudou [7,8], Koudou and Ley [9], Koudou and Vallois [10,11], Seshadri and Wesołowski [20]. Here we are interested in those which are extensions of the Matsumoto–Yor (MY) property of the univariate gamma and generalized inverse Gaussian distributions. The gamma $\gamma(p, a)$ and the generalized inverse Gaussian $GIG(q, b, c)$ distributions are defined by the densities

$$f(y) \propto y^{p-1} e^{-ay} I_{(0,+\infty)}(y)$$

and

$$g(x) \propto x^{q-1} e^{-bx-c/x} I_{(0,+\infty)}(x),$$

respectively, where p, a, b, c are positive numbers and q is real.

Matsumoto and Yor [17,18] considered the transformation ψ that takes $(x, y) \in (0, +\infty)^2$ into $(0, +\infty)^2$, where

$$\psi(x, y) = \left((x+y)^{-1}, x^{-1} - (x+y)^{-1}\right).$$

They observed that if two random variables X and Y are independent and follow the $GIG(-q, a, b)$ and $\gamma(q, a)$ distributions, respectively, then the two random variables U and V defined as $(U, V) = \psi(X, Y)$ are also independent and follow the $GIG(-q, b, a)$ and $\gamma(q, b)$ distributions, respectively. Letac and Wesołowski [14] proved the converse to the MY property, that is the following characterization: if X and Y are independent and U and V are also independent, where $(U, V) = \psi(X, Y)$, then $(X, Y) \sim GIG(-q, a, b) \otimes \gamma(q, a)$. In the same paper it was shown that this result holds true also for matrix variates, namely the authors considered the transformation ψ for X and Y positive definite random matrices and proved both the direct MY property and its converse in this case (under certain smoothness conditions, weakened later on in Wesołowski [21]): if X and Y are independent $r \times r$ positive definite matrices and $U = (X+Y)^{-1}$ and $V = X^{-1} - (X+Y)^{-1}$ are also independent then X and Y follow a matrix variate GIG and Wishart distribution, respectively.

For any $s \times r$ real matrix z of full rank, denote by $\mathbf{P}(z)$ the linear mapping

$$x \in \mathcal{V}_r \mapsto \mathbf{P}(z)x = zxz^t \in \mathcal{V}_s,$$

(z^t denotes the transpose of the matrix z). Massam and Wesołowski [16] extended the MY property to more general situation, where matrix variates have different dimensions: X and Y are independent positive definite matrices of dimensions $r \times r$ and $s \times s$, respectively. They considered the transformation ψ_z defined as follows

$$\psi_z(x, y) = \left((\mathbf{P}(z)x + y)^{-1}, x^{-1} - \mathbf{P}(z^t)(\mathbf{P}(z)x + y)^{-1}\right),$$

where z is a given constant $s \times r$ matrix of full rank and obtained the following characterization (under certain smoothness conditions): if X and Y are independent and U and V are also independent, where $(U, V) = \psi_z(X, Y)$, then X and Y follow a matrix variate GIG and Wishart distribution, respectively.

On the other hand, Massam and Wesołowski [15], interpreted the original MY property as a bivariate property with respect to the simple tree with two vertices and one edge and extended it to a p-variate property with respect to any tree with p vertices. Moreover, they proved the converse of this extended version of the MY property, obtaining the characterization of the product of one gamma and $p-1$ generalized inverse Gaussian distributions. To this end they considered certain transformations induced by leaves of such a tree.

In this paper we extend the multivariate version of the MY property on trees considered in [15] to the case where the components of a random vector corresponding to the vertices of the tree are replaced by random matrices of different dimensions. We prove this generalized MY property and its counter-part being a joint characterization of one Wishart and $p-1$ matrix generalized inverse Gaussian distributions.

The proof of our characterization is given under the assumption of strict positivity and differentiability of the densities. Here we do not use Laplace transforms to identify the Wishart variables (the Laplace transform approach was used in [15] to