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Univariate conditioning of vine copulas

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HIGHLIGHTS

- We construct and study a new subclass of vine copulas which are characterized by a star like first level tree—C*-vines.
- We study the interlink between tail dependence and univariate conditioning (truncation) of copulas.
- The formulas for tail dependence functions and limit copulas of the univariate conditioning are provided for C*-vines.
- The necessary and sufficient conditions for a C*-vine to be invariant with respect to univariate conditioning are presented.
- The possible applications to risk management are discussed.

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ABSTRACT

In this paper we deal with the tail behaviour of copulas. We compare the methods based on the univariate conditioning of a selected variable and on the tail dependence functions. We introduce a new subclass of vine copulas, consisting of regular vine copulas which are "rooted" at the first variable and study the limiting properties of such copulas.

1. Introduction

There are three ways of studying the tail behaviour of copulas at a given vertex v of the unit cube $[0, 1]^n$.

1. We restrict the given copula to a sufficiently small neighbourhood of the vertex v.
2. We restrict the given copula to a sufficiently small neighbourhood of a selected face of the cube containing the vertex v.
3. We restrict the given copula to a sufficiently small neighbourhood of a union of all faces of the cube containing the vertex v.

In all three cases, we shrink the neighbourhoods towards respectively the vertex, the face or the union of faces, and we study the limiting properties of copulas (compare [25–27,8] for the first approach, [28,29,13,21,37] for the second one, [4,5,33,34] for both first and second and [32,36,39] for the first and third one).

When the vertex v is the origin, $v = (0, \dots, 0)$, then the conditioning events 1, 2 and 3 can be described in terms of random variables and their α-quantiles, for sufficiently small α. Namely, if C is the copula of random variables X_i,

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$i = 1, \ldots, n$, then restricting of the copula C is equivalent to study the conditional dependence between $(X_1, \ldots, X_n)$ under the condition:

$$X_i \le q_\alpha(X_i) \tag{1}$$

for respectively:

(1′) all i;
(2′) a selected i;
(3′) any i.

For a vertex $v = (v_1, \ldots, v_n)$ other than the origin, to restate the conditioning events 1, 2 and 3 in terms of random variables one has to replace the condition (1) by

$$X_i \le q_\alpha(X_i) \quad \text{if } v_i = 0 \quad \text{or} \quad X_i \ge q_{1-\alpha}(X_i) \quad \text{if } v_i = 1. \tag{2}$$

In this paper we deal mostly with the second approach. We compare it with the first one and furthermore draw some consequences for vine copulas.

The motivation for the study follows from the financial and actuarial risk management, where the construction of appropriate models for dependence between risks is of obvious importance. Indeed, it is a well recognized fact that neglecting dependence gives rise to a dramatic risk underestimation. We illustrate the possible applications on two examples.

Example 1. Portfolio selection.

We study an investment portfolio with an important asset A_1. Let $[T_0, T_1]$ be an investment horizon. The crucial point is to determine what may happen with prices of other assets when there is a large loss caused by the leading one, i.e. to which extend the diversification could hedge the aggregated outcome.

Let the random variables $X_i,\ i = 1, \ldots, n$ model the prices of assets A_i at time T_1.

Problem S. Minimize the risk measure ρ of the portfolio, i.e. find the optimal structure of the portfolio based on some fixed set of strategies W:

$$\min_{\mathbf{w}\in W} \rho(w_1X_1 + w_2X_2 \cdots + w_nX_n | X_1 \le q),$$

where q is some threshold, for example 0.05 quantile of X_1.

Such situations as above, when one has to mitigate the losses in certain assets, say A_1, with the help of other investments, occur when for some reasons the position in A_1 cannot be closed or reduced. For example, because the A_1 shares are necessary to keep the controlling interest and assure the majority of voting stock shares to appoint the Board of Directors of the Limited Company A_1.

Example 2. Systemic risk in financial networks.

An intricate web of claims and obligations ties together the balance sheets of a wide variety of financial institutions, banks, hedge funds, and various intermediaries.

Due to the opinion of many researchers, these inter-bank claims have played a large role in the dissemination of the financial crisis of 2007–2008.

Let W_t^i be the measure of a welfare of the ith bank or financial institution at moment t.

When W_t^i falls below some fixed threshold, say ϵ close to 0, then a *default* takes place. Note that the default is not necessarily an extinction from the market, but might be for example:

a bailout,
a debt restructurization,
or a beginning of a recovery process.

Problem W. Determine the distribution of $W_t^1, \ldots, W_t^n$, when W_t^1 is in trouble, that is, the distribution of

$$\left(W_t^1, \ldots, W_t^n\right) \mid W_t^1 \le q.$$

So, to put it another way, we ask how systemic is the first institution [1,20,3] or how contagious [9–12,30]. The choice of q depends on the purpose for which the model is constructed. For example it might be such a threshold that when the welfare falls below it the bank supervision takes a "closer" look at the controlled institution. The exact measure of welfare depends on the employed default model. In "structural credit models" W_i will model the value of the ith firm's assets. In more practical approaches it might be Altman's Z-score formula or Capital Adequacy Ratio.

Note, that when we forget about the financial background the stated above W problem can be seen as a special case of a general problem of a limit law of a random vector with an extreme component, which was approached by other means in [22,23].