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Conditional quantiles and tail dependence

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ABSTRACT

Conditional quantile estimation is a crucial step in many statistical problems. For example, the recent work on systemic risk relies on estimating risk conditional on an institution being in distress or conditional on being in a crisis (Adrian and Brunnermeier, 2010; Brownlees and Engle, 2011). Specifically, the CoVaR systemic risk measure is based on a conditional quantile when one of the variable is in the tail of the distribution. In this paper, we study properties of conditional quantiles and how they relate to properties of the copula. In particular, we provide a new graphical characterization of tail dependence and intermediate tail dependence from plots of conditional quantiles with normalized marginal distributions (probit scale). A popular method to estimate conditional quantiles is the quantile regression (Koenker, 2005; Koenker and Bassett, 1978). We discuss the properties and pitfalls of this estimation approach.

1. Introduction

Conditional quantile estimation is a crucial step in many statistical problems and in quantitative risk management. For example, the recent work on systemic risk deals with estimating risk conditional on an institution or on the system being in distress. In this context, the conditional quantiles are conditional on some variables being in the tail of their distributions (see [2]). Other financial applications include [11] for robust beta estimation, [5] for macroeconomics risk, [27] for the relationship between expected value and volatility and [24,8] for estimating conditional Value-at-Risk, respectively systemic risk. In other fields, conditional quantiles have been used to estimate wage [9], economic growth [10], or educational attainment [19].

One contribution of this paper is to study how informative conditional quantiles are about asymmetric dependence, tail dependence [43] and intermediate tail dependence [30,12,31,13]. In particular, we propose a new graphical method to detect and estimate tail dependence by making use of the probit scale (for which marginal distributions are standard normal). It complements the recent work of Hua and Joe [32] and of Joe and Li [34]. Our approach deals with conditional quantiles. It is an alternative to the existing methods, which are related to the study of tail dependence coefficients [42], conditional expectations and strength of dependence in the tails [32] and it may also help to choose the appropriate dependence structure (or copula) [6,18].

Compared to the existing literature, our approach of tail dependence estimation is non symmetric, in the sense that our measure of tail dependence through conditional quantiles will depend on the order of the variables and thus on the non-exchangeability between the variables. It is possible that $X|Y$ displays tail dependence, whereas $Y|X$ does not display tail

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dependence. This is relevant in the context of studying tail dependence when the two variables under study do not play symmetric roles and are potentially of different order of magnitude. Recall that tail dependence refers typically to a property of the copula and therefore it does not take into account the marginal distributions and thus the order of magnitude of the variables. For instance, if X is the return on a market index and Y is the return on a small bank, then, a risk manager will be concerned of tail dependence of Y given X while having less interest in studying the tail dependence of X given Y. Potentially, in the context of systemic risk estimation, a regulator may be more interested in studying the tail dependence of X given Y, where X and Y represent the return of the financial market as a whole and of a big player respectively. We will show that the study of conditional quantiles is thus very insightful for studying tail dependence of non-exchangeable copulas, which standard tail dependence coefficients fail to measure adequately. We will illustrate this point by the study of the Tawn copula among others.

There are several methods to estimate conditional quantiles, including quantile regression [36,37], local quantile regression [46] and non parametric estimation of conditional quantiles [41]. Although quantile regression *per se* is only one of the possible approaches available to estimate a conditional quantile, it seems to be the most popular approach in recent studies that involve the estimation of conditional quantiles. The quantile regression method is a natural extension of the classical least squares estimation of conditional mean models to the estimation of the conditional quantile functions. In this paper, we explain what quantile regression consists of, why and when it can be used or should not be used. In particular, we give the conditions under which conditional quantiles are linear, and thus, under which (linear) quantile regression may be the appropriate tool to estimate a conditional quantile. An alternative approach can be found in [7] who propose to use non-linear quantile regression and design non-linear regressors linked to the dependence among the data.

In Section 2, we define "normalized conditional quantiles", i.e. conditional quantiles with standard normal margins (after transformation). We briefly describe the methodology behind quantile regression and illustrate graphically conditional quantiles for a wide range of possible dependence structure. This preliminary study already shows that conditional quantiles may be highly non-linear, and in particular that quantile regression fails to estimate conditional quantiles as soon as there is some tail dependence. Section 3 gives conditions under which conditional quantiles are linear and quantile regression is thus a suitable estimation approach. In Section 4, we then provide theoretical properties and explain these differences in shapes by properties of tail dependence and intermediate tail dependence.

2. Preliminary on conditional quantiles

Let $(X_1, X_2, \ldots, X_n, Y)$ be $n+1$ random variables. We are interested in the conditional distribution of Y given $\mathbf{X} := (X_1, X_2, \ldots, X_n)$. Let us denote by $F_{Y|\mathbf{X}=\mathbf{x}}$ the conditional cdf

$$F_{Y|\mathbf{X}=\mathbf{x}}(y) = P\left(Y \leqslant y | X_1 = x_1, \ldots, X_n = x_n\right)$$

and the conditional quantiles

$$F^{-1}_{Y|\mathbf{X}=\mathbf{x}}(\alpha) = \inf\left\{y \in \mathbb{R} \mid F_{Y|\mathbf{X}=\mathbf{x}}(y) \geqslant \alpha\right\}. \tag{1}$$

Throughout the paper, we assume that $X_1, X_2, \ldots, X_n, Y$ are continuously distributed.

2.1. Conditional quantiles for bivariate risks

Consider two continuously distributed risks X and Y. The conditional quantiles of Y given $X = x$ is given by $F^{-1}_{Y|X=x}(\alpha)$. From Sklar's theorem the joint distribution of X and Y is defined as

$$P(X \leqslant x, Y \leqslant y) = C(F_X(x), F_Y(y))$$

where F_X and F_Y denote the marginal distributions of X and Y respectively and where C denotes the copula between X and Y (unique as F_X and F_Y are continuous). Recall that the conditional probability of V given $U = u$ and U given $V = v$ can be computed by the first derivatives of the copula with respect to each of the variable. We denote by $C_{2|1}$ and $C_{1|2}$ these derivatives.[1] For example,

$$C_{2|1}(v|u) := P(V \leqslant v | U = u) = \frac{\partial}{\partial u} C(u, v),$$

and the conditional quantile of Y given $X = x$ at $\alpha \in (0, 1)$ is then given by

$$F^{-1}_{Y|X=x}(\alpha) = F_Y^{-1}\left(C^{-1}_{2|1}(\alpha | F_X(x))\right). \tag{2}$$

The notation $C^{-1}_{2|1}(\alpha|F_X(x))$ denotes the inverse of the function $v \mapsto C_{2|1}(v|F_X(x)) = \left(\frac{\partial C}{\partial u}(u, v)\right)_{u=F_X(x)}$.

It is clear that $F_Y(\cdot)$ has an important impact on the conditional quantile as it controls in particular the range of values taken by the conditional quantile. On the contrary, $F_X(\cdot)$ has no effect on the conditional quantile as $F_X(X)$ is uniformly

[1] $C_{2|1}(v|u)$ is also called the h function in the context of pair copula constructions. It is known explicitly for many bivariate copulas, see for example [1].