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Higher order tail densities of copulas and hidden regular variation

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ABSTRACT

A notion of higher order tail densities for copulas is introduced using multivariate regular variation of copula densities, and densities of multivariate extremes with various margins can then be studied in a unified fashion. We show that the tail of a multivariate density can be decomposed into the tail density of the underlying copula, coupled with marginal tail transforms of the three types: Fréchet, Gumbel, and Weibull types. We also derive the relation between the tail density and tail order functions of a copula in the context of hidden regular variation. Some illustrative examples are given.

1. Introduction

To facilitate tail inference with various multivariate distributions, a better understanding of the strength of dependence in joint lower or joint upper distribution tails is often needed. In particular, in this paper we are interested in analyzing scale-invariant tail dependence strength of a multivariate distribution, separated from its univariate margins; that is, we are interested in analyzing tail dependence via the copula approach.

The main purpose of this paper is to develop a general copula tail density approach, so that tail properties can be derived directly from joint tails of multivariate densities. Most multivariate distributions are specified by densities and the tail density approach is especially tractable when a multivariate density has a simple, explicit expression, whereas its joint cumulative distribution function does not have a closed form. This research is motivated by the need to analyze the tail risk measures that are often expressed in terms of tail densities of the multivariate copulas of underlying loss distributions [12,25].

Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a random vector with distribution F and continuous marginal distributions $F_1, \ldots, F_d$. Let $\overline{F}$, and $\overline{F}_1, \ldots, \overline{F}_d$ denote the corresponding survival functions. Assume throughout this paper that F has a density function f. The tail behavior of F or f is often described using the notion of multivariate regular variation [22,24], and without loss of generality, we assume that F concentrates on $\mathbb{R}_+^d = [0, \infty)^d$. A univariate Borel-measurable function $V : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be regularly varying at ∞ with tail index $\rho \in \mathbb{R}$, denoted by $V \in \mathrm{RV}_\rho$, if $V(tx)/V(t) \to x^\rho$ as $t \to \infty$ for any $x > 0$. A d-dimensional density function f is said to be (multivariate) regularly varying at ∞ with a limiting function $\lambda(\cdot)$ if

$$\lim_{t\to\infty} \frac{f(t\boldsymbol{x})}{t^{-d}V(t)} = \lambda(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}_+^d \setminus \{\boldsymbol{0}\}, \tag{1.1}$$

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for some univariate regularly varying function $V \in \mathrm{RV}_{-\alpha}$ where $\alpha > 0$. The tail density $\lambda(\cdot)$ in (1.1) was introduced in [2], and de Haan and Resnick proved in [3] that if furthermore (1.1) converges uniformly on the unit sphere of $\mathbb{R}_+^d$, then the regular variation (1.1) of a density implies multivariate regular variation of its cumulative distribution function F; i.e.,

$$\lim_{t\to\infty} \frac{1-F(t\boldsymbol{x})}{V(t)} = \int_{[\mathbf{0},\boldsymbol{x}]^c} \lambda(\boldsymbol{v})d\boldsymbol{v}, \quad \boldsymbol{x} > \mathbf{0}, \tag{1.2}$$

where $[\mathbf{0}, \boldsymbol{x}]^c = \left(\prod_{i=1}^d [0, x_i]\right)^c$ denotes the complement of $\prod_{i=1}^d [0, x_i]$ for $\boldsymbol{x} = (x_1, \ldots, x_d)$. Here and in the sequel, vector addition/product and vector inequalities are operated component-wise. The fact that (1.1) implies (1.2) can be viewed as a multivariate extension of Karamata's theorem for univariate regular variation. In contrast to the univariate case, however, the uniform convergence on the unit sphere of $\mathbb{R}_+^d$ is needed in the multivariate case to control the function's variation moving from ray to ray originated from $\mathbf{0}$. There are multivariate distributions that satisfy (1.1) but not (1.2) (see [3]). The regular variation property (1.2) is crucial in deriving multivariate extreme value distributions for random samples drawn from distribution F [24].

The scale-invariant tail behavior of a multivariate distribution F can be studied using its marginally transformed distribution $F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))$, known as the copula of F [11]. A copula is invariant under marginal increasing transforms and thus preserves the scale-invariant dependence structure of the distribution. The strongest form of scale-invariant dependence in the distribution tails is the first-order tail dependence when as $u \to 0^+$, $F(F_1^{-1}(u), \ldots, F_d^{-1}(u)) \sim u$ in the lower tail and $\overline{F}(\overline{F}_1^{-1}(u), \ldots, \overline{F}_d^{-1}(u)) \sim u$ in the upper tail, which are also studied under the notion of tail comonotonicity in [7,8]; see [11,14,15] for more details about the first-order tail dependence that is simply referred to as tail dependence in the literature. Here and in the sequel, for any invertible function $h(\cdot)$, $h^{-1}(\cdot)$ denotes its inverse, and the tail equivalence of two functions $f(x) \sim g(x)$ as $x \to a$, $a \in \overline{\mathbb{R}}$, means that $\lim_{x\to a}[f(x)/g(x)] = 1$. The strength of higher order scale-invariant tail dependence can be characterized via the lower and upper tail orders $\kappa_L, \kappa_U \geq 1$ (see [10,6]) when

$$F(F_1^{-1}(u), \ldots, F_d^{-1}(u)) \sim u^{\kappa_L}\ell_L(u), \quad u \to 0^+, \tag{1.3}$$

in the lower tail and

$$\overline{F}(\overline{F}_1^{-1}(u), \ldots, \overline{F}_d^{-1}(u)) \sim u^{\kappa_U}\ell_U(u), \quad u \to 0^+, \tag{1.4}$$

in the upper tail, where $\ell_L(\cdot)$, and $\ell_U(\cdot)$ are slowly varying functions at 0 (i.e., $\ell_L(1/t), \ell_U(1/t) \in \mathrm{RV}_0$). Smaller values of κ_L (κ_U) indicate stronger dependence in the joint lower (upper) tail, and in contrast to the first-order tail dependence, there can be intermediate tail dependence when the tail order is between 1 and d. The tail orders are easy to compute when the distributions and quantile functions have closed forms [6], and it becomes difficult to compute for the distributions that are only specified by their densities. A copula tail density approach was developed in [18] to study the first-order scale-invariant tail dependence of multivariate distributions with tractable densities. Furthermore, it was shown in [18] (also see [16]) that the tail density $\lambda(\cdot)$ in the case of multivariate regular variation (1.2) can be decomposed into the copula tail density and marginal power transforms. In this paper, we extend the tail density approach to analyze the scale-invariant tail dependence and tail order for copulas that are specified only by their densities. Specifically, we show that the tails of various multivariate densities can be written in terms of higher order copula tail densities and marginal tail transforms of the three types (Fréchet, Gumbel, and Weibull). We also show that under mild regularity conditions, regular variation of tail densities of copulas, together with regularly varying margins, imply hidden regular variation (HRV); that is, multivariate regular variation resided within the interior of $\mathbb{R}_+^d$ where the joint tail probability decays to zero faster than marginal univariate regular variation.

We introduce in Section 2 the higher order tail density of a copula and apply it to analyze the tails of multivariate densities. We prove in Section 3 a multivariate copula version of Karamata's theorem for the distributions with hidden regular variation. Some remarks in Section 4 conclude the paper.

2. Tail densities of copulas

A copula C is a multivariate distribution with uniformly distributed univariate margins on [0, 1]. Sklar's theorem (see, e.g., Section 1.6 in [11]) states that every multivariate distribution F with margins $F_1, \ldots, F_d$ can be written as $F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$ for some d-dimensional copula C. In fact, in the case of continuous univariate margins, C is unique and

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)).$$

Let $(U_1, \ldots, U_d)$ denote a random vector with distribution C and U_i, $1 \leq i \leq d$, being uniformly distributed on [0, 1]. The survival copula $\widehat{C}$ is defined as follows:

$$\widehat{C}(u_1, \ldots, u_d) = \mathbb{P}(1-U_1 \leq u_1, \ldots, 1-U_d \leq u_d) = \overline{C}(1-u_1, \ldots, 1-u_d) \tag{2.1}$$

where $\overline{C}$ is the joint survival function of C. The survival copula $\widehat{C}$ can be used to transform lower tail properties of $(U_1, \ldots, U_d)$ into the corresponding upper tail properties of $(1-U_1, \ldots, 1-U_d)$. Assume throughout this paper that the density $c(\cdot)$ of copula C exists, and that $c(\cdot)$ is continuous in some small open neighborhoods of $\mathbf{0}$ and $\mathbf{1} = (1, \ldots, 1)$ (i.e., ultimately continuous at $\mathbf{0}$ and $\mathbf{1}$).