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Extremes of scale mixtures of multivariate time series

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ABSTRACT

Factor models have large potential in the modeling of several natural and human phenomena. In this paper we consider a multivariate time series $\mathbf{Y}_n, n \geq 1$, rescaled through random factors $\mathbf{T}_n, n \geq 1$, extending some scale mixture models in the literature. We analyze its extremal behavior by deriving the maximum domain of attraction and the multivariate extremal index, which leads to new ways to construct multivariate extreme value distributions. The computation of the multivariate extremal index and the characterization of the tail dependence show an interesting property of these models. More precisely, however much it is the dependence within and between factors $\mathbf{T}_n, n \geq 1$, the extremal index of the model is unit whenever $\mathbf{Y}_n, n \geq 1$, presents cross-sectional and sequential tail independence. We illustrate with examples of thinned multivariate time series and multivariate autoregressive processes with random coefficients. An application of these latter to financial data is presented at the end.

1. Introduction

Factor models have been used in the modeling of data within hydrology (Nadarajah [26,27] 2006/2009, Nadarajah and Masoom [28] 2008), storm insurance (Lescourret and Robert, [22] 2006), soil erosion in crops (Todorovic and Gani [36] 1987, Alpuim and Athayde [2] 1990), reliability (Alpuim and Athayde [2] 1990, Kotz et al. [19] 2000), economy (Arnold, [3] 1983) and finance (Ferreira and Canto e Castro, [13] 2010).

Let $\mathbf{X}_n = (X_{n1}, \ldots, X_{nd}),\ n \geq 1$, be a d-variate sequence, such that $X_{nj} = Y_{nj}T_{nj},\ j = 1, \ldots, d$, where

(a) $\mathbf{Y} = \{(Y_{n1}, \ldots, Y_{nd})\}_{n\geq 1}$ is a stationary sequence such that, Y_{nj} has a Pareto-type distribution $F_{Y_j},\ j = 1, \ldots, d$, i.e., for each $j = 1, \ldots, d$, there exists a positive constant β_j for which

$$F_{Y_j}(x) = 1 - x^{-\beta_j} l_{Y_j}(x), \tag{1}$$

with l_{Y_j} a slowly varying function, i.e., $l_{Y_j}(ax)/l_{Y_j}(x) \to 1$, as $x \to \infty$, for all $a > 0$,

(b) $\mathbf{T} = \{(T_{n1}, \ldots, T_{nd})\}_{n\geq 1}$ is a stationary sequence, independent of $\mathbf{Y}$, with support $\mathbb{R}^d_+$ and such that $E(T_{nj}^{\epsilon_j}) < \infty$, for some $\epsilon_j > \beta_j,\ j = 1, \ldots, d$.

This work is concerned with the extremal behavior of the multivariate time series $\mathbf{X}_n$, extending most of the factor models mentioned above. More precisely, we derive the max-domain of attraction (Section 2), calculate the multivariate extremal index (Section 3) and characterize the tail dependence (Section 4).

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The product $Y_{nj}T_{nj}$ can be seen as a random normalization of Y_{nj} by T_{nj}, which is often required when modeling extremal behavior. For instance, if Y_{nj} is the rate of an extreme event and T_{nj} its average cost, then $Y_{nj}T_{nj}$ can be interpreted as the total cost of the extreme event. Products of two independent random variables where one of them is regularly varying have been addressed from both theoretical and applied points of view (Maulik et al. [24] 2002, Lescourret and Robert [22] 2006, Nadarajah [26] 2006 and references therein).

Our motivation to the probabilistic study of extremes of multivariate sequences of products was originated from some particular models. Consider, for instance that T_{nj} are Bernoulli distributed. Then $\mathbf{X}_n$ provides a model for multivariate data subjected to missing values. Extremes of univariate sequences with random missing values have been considered in Weissman and Cohen ([37], 1995) as a particular case of some mixture models. Additional results on extremes of incomplete samples can be found in Mladenovic and Piterbarg ([25], 2006) and Tan and Wang ([34], 2012).

Li ([23], 2009) analyzed the tail dependence of the scale mixture $\mathbf{X}_n$ when $\mathbf{Y}_n = (Y_{n1}, \ldots, Y_{nd})$ has multivariate extreme value distribution with standard Fréchet margins and $T_{nj} = T_n, j = 1, \ldots, d$. Here we consider scale mixtures of multivariate sequences which are very flexible models for data exhibiting tail dependence and asymptotic tail independence such as, respectively, ARMAX and pARMAX sequences (Ferreira and Ferreira [14]). We give particular emphasis to a model in which $\beta_j = \alpha/\gamma_j, \alpha, \gamma_j > 0,\ j = 1, \ldots, d$, generalizing the results of Lescourret and Robert ([22], 2006) (Section 5. An application to financial data will be provided at the end (Section 6).

2. Preliminary results and max-domain of attraction

We start with some properties of $\{\mathbf{X}_n\}_n \geq 1$, that will be used along the paper. We use notation $r_j = E(T_{nj}^{\beta_j})$ along the paper.

Proposition 2.1. *For each $j = 1, \ldots, d$, $\{X_{nj}\}_{n\geq 1}$ is a stationary sequence having Pareto-type distribution.*

Proof. First, observe that

$$\lim_{x\to\infty} \frac{P(Y_{nj}T_{nj} > x)}{P(Y_{nj} > x)} = \lim_{x\to\infty} \frac{\int x^{-\beta_j} z^{\beta_j} l_{Y_j}(x/z) dP_{T_{nj}}(z)}{x^{-\beta_j} l_{Y_j}(x)} = r_j,$$

where the last step is due to the dominated convergence theorem and by using the Potter bounds of regularly varying functions (Bingham et al., [5] 1987; Theorem 1.5.6.). Therefore, for large x,

$$1 - F_{X_j}(x) = P(X_{nj} > x) = x^{-\beta_j} l_{Y_j}(x) r_j (1 + o(1)) := x^{-\beta_j} l_{X_j}(x), \tag{2}$$

where it is immediately seen that l_{X_j} is a slowly varying function. □

In the sequel we denote $U_{X_j}(x)$ and $U_{Y_j}(x)$ the quantile functions, $F_{X_j}^{-1}(1 - 1/x)$ and $F_{Y_j}^{-1}(1 - 1/x)$, respectively.

Given (1) and applying the Bruyn conjugate concept (Beirlant et al. [4] 2004, Proposition 2.5), we have that, for large x,

$$U_{Y_j}(x) = x^{1/\beta_j} l_{U_{Y_j}}(x) = x^{1/\beta_j} l_{Y_j}^{1/\beta_j}(x^{1/\beta_j})(1 + o(1)), \tag{3}$$

where $l_{U_{Y_j}}$ is a slowly varying function. Using again the Bruyn conjugate concept and by (2), we can state, for large x,

$$U_{X_j}(x) = x^{1/\beta_j} l_{U_{X_j}}(x) = x^{1/\beta_j} l_{X_j}^{1/\beta_j}(x^{1/\beta_j})(1 + o(1)) = x^{1/\beta_j} \left(l_{Y_j}(x^{1/\beta_j}) r_j\right)^{1/\beta_j} (1 + o(1)),$$

where $l_{U_{X_j}}$ is a slowly varying function. Therefore, and considering (3), we have for large x,

$$U_{X_j}(x) = x^{1/\beta_j} l_{U_{Y_j}}(x) r_j^{1/\beta_j} (1 + o(1)) = U_{Y_j}\left(r_j x\right)(1 + o(1)). \tag{4}$$

Proposition 2.2. *The upper tail copula function of $\mathbf{X}$ is given by*

$$\Lambda_{\mathbf{X}}(x_1, \ldots, x_d) = E\left(\Lambda_{\mathbf{Y}}\left(\frac{T_1^{\beta_1} x_1}{r_1}, \ldots, \frac{T_d^{\beta_d} x_d}{r_d}\right)\right),$$

with $(x_1, \ldots, x_d) \in \overline{\mathbb{R}}_+^d = [0, \infty]^d \setminus \{(\infty, \ldots, \infty)\}$, where $\mathbf{T} = (T_1, \ldots, T_d)$ is a random vector distributed as $\mathbf{T}_n = (T_{n1}, \ldots, T_{nd})$ and provided that the upper tail copula function of $\mathbf{Y}_n$ exists, i.e., the limit

$$\Lambda_{\mathbf{Y}}(x_1, \ldots, x_d) = \lim_{t\to\infty} tP\left(\bigcap_{j=1}^{d} \left\{Y_{1j} > U_{Y_j}\left(t/x_j\right)\right\}\right) \tag{5}$$

is finite.