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Goodness-of-fit tests for multivariate stable distributions based on the empirical characteristic function

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ABSTRACT

We consider goodness-of-fit testing for multivariate stable distributions. The proposed test statistics exploit a characterizing property of the characteristic function of these distributions and are consistent under some conditions. The asymptotic distribution is derived under the null hypothesis as well as under local alternatives. Conditions for an asymptotic null distribution free of parameters and for affine invariance are provided. Computational issues are discussed in detail and simulations show that with proper choice of the user parameters involved, the new tests lead to powerful omnibus procedures for the problem at hand.

1. Introduction

Let X be a p-variate ($p \geq 1$) random vector and $\varphi(t)$ denote its characteristic function (CF). It is well known (see [35, Eq. (13.1)]) that X follows a multivariate stable distribution if for any $a > 0$, there are $b > 0$ and $c \in \mathbb{R}^p$ such that

$$(\varphi(t))^a = \varphi(bt)e^{it'c} \quad t \in \mathbb{R}^p. \tag{1.1}$$

The law of X is called strictly stable if (1.1) holds with $c = 0$. In this paper relation (1.1) will be exploited to construct goodness-of-fit tests for multivariate stable distributions with special attention devoted to tests for symmetric stable distributions, and to Cauchy and normal distributions in particular.

Previous related works closely connected to the approach followed here are those of Csörgő [6], Henze and Zirkler [15], Henze and Wagner [14], Epps [10], and Pudelko [31], for testing multivariate normality. Hušková and Meintanis (2007) [16] and Jiménez-Gamero et al. [18] extend this approach in order to test for the distribution of random errors in the context of linear regression models, with special emphasis on testing normality. The aforementioned tests though should certainly not be confined to testing for normality. In fact they are in principle meant for general use. Nevertheless, the underlying approach cannot be readily applied to arbitrary distributions. The reason is that in the test statistic, empirical

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and parametric CFs are directly compared by means of a distance function which is not always easy to compute; see e.g., [17]. This drawback is clearly evident in the case of testing for the Cauchy distribution. Specifically, while this direct approach was straightforward to adapt by Gürtler and Henze [12] and Matsui and Takemura [25] in order to test goodness-of-fit to this particular distribution, the generalization attempted by Matsui and Takemura [26] in order to test for an arbitrary univariate symmetric stable distribution requires numerical integration. Here we circumvent this problem by using (1.1) in the construction of the test statistic, which characterizes the family of multivariate stable distributions. Note that the characterization approach is also followed by Arcones [1] in the special case of testing for the normal distribution. This approach leads to the clear advantage of computational simplicity, which is a major issue particularly in the multivariate setting where high dimensional integration is often troublesome. Also, depending on the case at hand, one can avoid estimating a location parameter which implies that in the context of composite goodness-of-fit testing we only need to estimate the covariance (or scatter) matrix. Moreover, the characterization approach involves appropriate choices of the parameters a and b in (1.1) which provides some flexibility in that it allows one to obtain extremely powerful tests for a large range of alternatives, by appropriately choosing the values of these user-specified parameters.

The structure of the paper is as follows. In Section 2 we will recall some basic features of the CF of multivariate stable distributions and discuss the characterization in (1.1). In Section 3 the test statistics are introduced, the case of testing for the normal and Cauchy distribution are emphasized, and the asymptotic properties of the proposed test statistic are derived. Section 4 proposes an affine-invariant version of the new test statistic, presents computational strategies, and analyzes the effect of user-specified parameters. In Section 5 we present simulation results on the power of the tests. Some conclusions and discussion are contained in Section 6. Several technical arguments are collected in an Appendix.

2. Multivariate stable distributions

First we introduce some basic notation and abbreviations. The identity matrix of dimension $p \times p$ will be denoted by $\mathrm{I_p}$. Also, we will indicate by $|A|$ the determinant of a matrix A. The same notation will be used for the norm $|x| = \left(\sum_{j=1}^{p} x_j^2\right)^{1/2}$ of a vector $x = (x_1, \dots, x_p)'$, while for the inner product of two vectors x, y we will write $x'y$. We shall write $o_P(1)$ to denote a quantity converging in probability to 0. Finally 'independent and identically distributed' will be abbreviated to i.i.d. Further, more specialized, notation will be introduced in Section 3.

The analytic study of stable, and more generally of infinitely divisible, distributions dates back to Lévy [23] and Feldheim [11] and makes use of characterization (1.1) and other equivalent statements. More recently Press [30] obtained the log—CF of a stable distribution as

$$\log \varphi(t) = i\delta' t - \frac{1}{2}\sum_{j=1}^{m} \left(t'\Sigma_j t\right)^{\alpha/2} \left[1 + i\beta(t;\alpha)\right] \tag{2.1}$$

with

$$\beta(t;\alpha) = \begin{cases} -\tan\left(\dfrac{\pi\alpha}{2}\right) \dfrac{\sum_{j=1}^{m} \left(t'\Sigma_j t\right)^{\alpha/2} \frac{w_j' t}{|w_j' t|}}{\sum_{j=1}^{m} \left(t'\Sigma_j t\right)^{\alpha/2}} & \text{if } \alpha \neq 1, \\[2ex] \dfrac{2}{\pi} \dfrac{\sum_{j=1}^{m} \left(t'\Sigma_j t\right)^{1/2} \frac{w_j' t}{|w_j' t|} \log |w_j' t|}{\sum_{j=1}^{m} \left(t'\Sigma_j t\right)^{1/2}} & \text{if } \alpha = 1, \end{cases}$$

where $0 < \alpha \le 2$, Σ_j is a positive definite matrix of rank r_j, $1 \le r_j \le p$, $j = 1, 2, \dots, m$ and no two Σ_j's are proportional; for further details and derivation of the formula see [30].

By imposing a symmetry condition around a location vector $\delta \in \mathbb{R}^p$, i.e. requiring that $\varphi(t)$ satisfies $e^{-i\delta' t}\varphi(t) = e^{i\delta' t}\varphi(-t)$ for all $t \in \mathbb{R}^p$, implies that $\beta = 0$, identically in t; thus a multivariate stable distribution symmetric around δ has $\log \varphi(t) = i\delta' t - \frac{1}{2}\sum_{j=1}^{m} \left(t'\Sigma_j t\right)^{\alpha/2}$, which may also be parametrized in a slightly different way as

$$\varphi(t) = e^{i\delta' t - (t'\Sigma t)^{\alpha/2}}, \quad t \in \mathbb{R}^p, \tag{2.2}$$

where Σ is assumed to be positive definite; see also [34, Section 5.2].

If $\alpha = 2$, then we have from (2.2) the CF of a multivariate normal distribution with mean vector δ and covariance matrix 2Σ. The case of the multivariate Cauchy distribution symmetric around δ is given by (2.2) when $\alpha = 1$. In this last case as well as for all $\alpha \in (0, 2)$, the matrix Σ will be understood as a general *scatter* matrix.

From the results above it follows that definition (1.1) is characteristic of general multivariate stable distributions. In particular, for the normal and Cauchy cases which we will consider in more detail we state the following propositions which are easily verified.