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A test for using the sum score to obtain a stochastic ordering of subjects

R. Ligtvoet *

Faculty of Social and Behavioural Sciences, University of Amsterdam, The Netherlands

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ABSTRACT

For many psychological test applications, the simple sum score across the items is used to make inferences about subjects. However, most of the item response theory models for psychological test data do not support such usage of the sum score. A simple test is proposed to assess whether the sum score can be used to obtain a stochastic ordering of subjects. This test is based on general (nonparametric) conditions and requires only the estimation of the unconstrained proportions.

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1. Introduction

Psychological tests usually consist of multiple items that are assumed to elicit responses that are considered typical for the attribute that the test is suppose to measure. Consider $X = (X_1, \ldots, X_k)$ to be the score vector containing the random item-score variables X_i $(i = 1, \ldots, k)$, and let $x_i \in \{1, \ldots, m_i\}$ be the item score obtained by assigning a score to the response of a subject to the item. In item response theory, a real-valued latent variable Θ is introduced to account for the associations that exist between the item scores, and the aim is to align subjects on Θ based on their score vector x; see Sijtsma and Junker [23] for an overview. Although many models have been proposed for this purpose, in practice, the ordering of subjects is most often basis on the simple sum score $X_+ = \sum_{i=1}^{k} X_i$. For the Rasch model [15,18], this sum score is a sufficient statistics for the latent variable, but the restrictive parametric requirements the Rasch model imposes on the data also means that the model may not be appropriate for the data. Rejection of the Rasch model, however, does not imply that the sum score cannot be used to stochastically order subjects on Θ. For most other models for psychological test data (other than binary data), there exists no basis to expect a stochastic ordering of Θ by X_+ [6,7,26]. Hence, these models do not provide information on whether the sum score can be used or not. These latter models include Molenaar's double monotonicity model [16], Scheiblechner's isotonic ordinal probabilistic model [22], Samejima's graded response model [19] and acceleration model [20], and Tutz's sequential Rasch model [25].

Ligtvoet [14] derived nonparametric conditions that are sufficient for obtaining a stochastic ordering of Θ by X_+. In Section 2, it is shown that the same conditions also imply that the sums of subsets of items from the test are totally positive of order 2 (TP2) [11]. This TP2 property is used in Section 3 as a basis for constructing a test for the conditions for using X_+ to order subjects on Θ. Some practical considerations for dealing with sparse data are considered, followed by a practical application of the test procedure.

* Correspondence to: Nieuwe Achtergracht 127, 1018 WS, Amsterdam, The Netherlands.
E-mail address: r.ligtvoet@uva.nl.
URL: https://www.sites.google.com/site/rligtv/.

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2. Preliminary results

In this section, some TP2 properties and results are discussed, and a new TP2 property is considered which is based on the sum score of disjoint subsets of items from the test. The main result provides sufficient conditions for the new TP2 property.

Definition 1. If two item-score variables X_1 and X_2 have a joint probability mass function $f_{X_1,X_2}(x_1, x_2)$, where for all $x_1 < m_1$ and $x_2 < m_2$,

$$f_{X_1,X_2}(x_1, x_2)f_{X_1,X_2}(x_1 + 1, x_2 + 1) \geq f_{X_1,X_2}(x_1, x_2 + 1)f_{X_1,X_2}(x_1 + 1, x_2),$$

then function f_{X_1,X_2} is said to be totally positive of order 2 (TP2).

The property TP2 implies a strong form of positive dependence between the variables, but without relying on parametric assumptions, like (log-) linearity [4]. Also, TP2 corresponds to a likelihood ratio ordering, where $f_{X_1}(x_1+1 \mid X_2 = x_2)/f_{X_1}(x_1 \mid X_2 = x_2)$ is nondecreasing in x_2, allowing X_1 to be stochastically ordered by X_2 [13], and visa versa.

Next, the TP2 property is generalized to account for more than two items. Define the function $f_X(x)$ for the $\prod_{i=1}^{k} m_i$ configurations of $x = (x_1, \ldots, x_k)$, and let for two vectors x and x^* the element-wise minimum and maximum be denoted by $\min(x, x^*)$ and $\max(x, x^*)$, respectively. Then, $f_X(x)$ is multivariate TP2, if $f_X\{\min(x, x^*)\}f_X\{\max(x, x^*)\} \geq f_X(x)f_X(x^*)$, for all x and x^* [12]. It is convenient to express multivariate TP2 in terms of TP2 for two items, conditional on the scores on the remaining items. Let $X = \{(X_i, X_j), Z\}$ denote the partition of X, whereby Z contains the scores on all items, except for items i and j.

Definition 2. The function $f_X(x)$ is said to be multivariate TP2 if the function $f_{X_i,X_j}(x_i, x_j \mid Z = z)$ is TP2, for all $i, j \in \{1, \ldots, k\}$ [12, p. 469].

The property of multivariate TP2 is important in the context of item response theory. In item response theory, assumptions are generally made on the conditional distribution of the item-score vectors X by a randomly chosen subject, for which $f_X(x) = \int f_X(x|\Theta = \theta)dF(\theta)$ [10]. Assume throughout that $f_{X_i}(x_i \mid \Theta = \theta)$ is strictly positive for all item scores x_i. Holland and Rosenbaum [8, Theorem 5] showed that, for any pair of items i and j, the function $f_{X_i,X_j}(x_i, x_j \mid h(Z) = r)$ is TP2 for any measurable function h, if the following two conditions hold

(i) conditional independence: $f_X(x \mid \Theta = \theta) = \prod_{i=1}^{k} f_{X_i}(x_i \mid \Theta = \theta)$,
(ii) and $f_{X_i}(x_i \mid \Theta = \theta)$ as a function of x_i and θ, is TP2, for all i.

Holland and Rosenbaum [8] refer to condition (ii) as latent TP2, and the property that $f_{X_i,X_j}(x_i, x_j \mid h(Z) = r)$ is TP2 is referred to as conditional multivariate TP2. Their result allows a range of models that imply the two conditions to be tested, by inspecting whether conditional multivariate TP2 holds for the observable item score distribution.

For the main result of this paper, a third condition is needed. This condition is similar to the definition of log-concave sequences [24], but applied to function $f_{X_i}(x_i \mid \Theta = \theta)$.

Definition 3. For $m_i > 2$, function $f_{X_i}(x_i \mid \Theta = \theta)$ is said to be log-concave, if for all $x_i < m_i - 1$ it holds that: $\{f_{X_i}(x_i + 1 \mid \Theta = \theta)\}^2 \geq f_{X_i}(x_i \mid \Theta = \theta)f_{X_i}(x_i + 2 \mid \Theta = \theta)$.

For $m_i = 2$, log-concavity holds by definition. This definition leads to the following condition:

(iii) $f_{X_i}(x_i \mid \Theta = \theta)$ is log-concave, for all items i.

Finally, let $X = (A, B, D)$ be a partition of X, where A and B contain at least one item-score variable. Also, let A_+ denote the sum of the item-score variables in A, and arbitrarily assign rank numbers to the outcomes of A_+, such that its realization $a \in \{1, \ldots, m_a\}$, where m_a corresponds to the maximum rank number assigned to A_+. Let B_+ and D_+, with realizations b and d, respectively, be similarly defined. The main result of this paper is summarized in the following theorem.

Theorem 1 (*Main Result*)**.** *Function $f_{A_+,B_+}(a, b \mid h(D) = s)$ is TP2 for any partition $X = (A, B, D)$, if in addition to conditions* (i) *and* (ii)*, also condition* (iii) *holds.*

Proof. The conditions (i), (ii), and (iii) imply that $f_{X_+,\Theta}(x_+, \theta)$ is TP2 [14, Theorem]. Because this result holds for any subset of items from $\{1, \ldots, k\}$, the three conditions also imply for A and B that $f_{A_+,\Theta}(a, \theta)$ and $f_{B_+,\Theta}(b, \theta)$ are both TP2. Also, $f_{A_+}(a \mid \Theta = \theta)$ and $f_{B_+}(b \mid \Theta = \theta)$ are conditionally independent (condition *i*). So, by replacing Y in the proof Holland and Rosenbaum [8, Theorem 5] with (A_+, B_+) and replacing Z by D, the desired result is obtained.

Ligtvoet [14] called the model defined by the three conditions the isotonic partial credit model, and he showed that it implies that $f_{X_+,\Theta}(x_+, \theta)$ is TP2. Hence, the isotonic partial credit model provides a theoretical justification for stochastically ordering subjects on Θ by their sum scores X_+, without resorting to parametric assumptions.

3. Test procedure

In order to test the conditions of Theorem 1, the vector X is split into three parts A, B, and D, for which we inspect whether $f_{A_+,B_+}(a, b \mid h(D) = s)$ is TP2. For practical reasons, the sum score D_+ is chosen for $h(D)$, and TP2 is assessed three times, with each set once taking on the role for conditioning variable.

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